9.3 Geometric Sequences and Series

Objectives

- Recognize, write, and find the *n*th terms of geometric sequences.
- Find the sum of a finite geometric sequence.
- Find the sum of an infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Geometric Sequences

Geometric Sequences

We have learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence.

In this section, you will study another important type of sequence called a **geometric sequence.**

Consecutive terms of a geometric sequence have a common *ratio*.

Geometric Sequences

Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric when there is a number r such that

r = common ratio

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \quad r \neq 0.$$

The number r is the **common ratio** of the geometric sequence.

1

A geometric sequence may be thought of as an exponential function whose domain is the set of natural numbers.

$2^{1-1} = 2^0 = 1$

$5, 10, 20, 40$

$1 \quad 2 \quad 3 \quad 4$

$5 \cdot 1 \quad 5 \cdot 2^1 \quad 5 \cdot 2^2 \quad 5 \cdot 2^{n-1}$

$1, 2, 4, 8, 16, \ldots, 2^{n-1}$

$1 \quad 2 \quad 3 \quad 4 \quad 5$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \qquad 2^n$

6

Example 1 Write the first 4 terms.

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio of consecutive terms is $r = 2$.

1a

$2^1, 2^2, 2^3, 2^4$

$2, 4, 8, 16$

$r = \frac{4}{2} = 2$

b. The sequence whose nth term is $4(3^n)$ is geometric. The common ratio of consecutive terms is $r = 3$.

1b

$4(3^1), 4(3)^2, 4(3)^3, 4(3^4)$

$12, 36, 108, 324$

$r = \frac{36}{12} = 3$

c. The sequence whose nth term is $(-1/3)^n$ is geometric. For this sequence, the common ratio of consecutive terms is $r = -\frac{1}{3}$.

1c

$(-\frac{1}{3})^1, (-\frac{1}{3})^2, (-\frac{1}{3})^3, (-\frac{1}{3})^4$

$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}$

$r = \frac{\frac{1}{9}}{-\frac{1}{3}} = \frac{1}{9} \cdot \frac{-3}{1} = \frac{-3}{9} = -\frac{1}{3}$

Reminders about Arithmetic Sequences and Series:

$a_n = a_1 + (n-1)d$

recursive

$a_1 = \text{given}$

$a_n = a_{n-1} + d$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d)$

Geometric Sequences

The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots\ldots,\ a_n,\ \ldots\ldots$$

$$a_1,\ a_1 r,\ a_1 r^2,\ a_1 r^3,\ a_1 r^4,\ \ldots,\ a_1 r^{n-1},\ \ldots$$

$a_n = a_1 r^{n-1}$

When you know the nth term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is, $a_{n+1} = a_n r$.

Geometric

$a_n = a_1 r^{n-1}$

recursive

$a_1 =$ given

$a_{n+1} = a_n r$

Example 2
Write the first five terms of the geometric sequence whose first term is $a_1 = 3$ and whose common ratio is $r = 2$. Then graph the terms on a set of coordinate axes.

(2) $a_1 = 3$ $r = 2$

$a_2 = 3 \cdot 2 = 6$

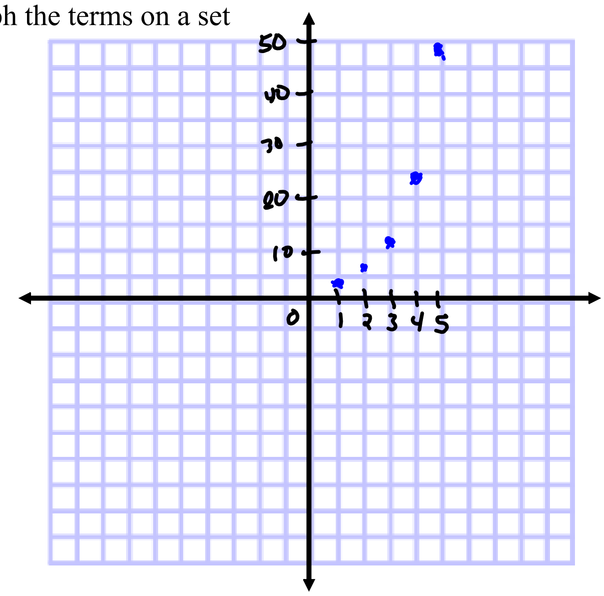
$a_3 = 6 \cdot 2 = 12$

$a_4 = 12 \cdot 2 = 24$

$a_5 = 24 \cdot 2 = 48$

3, 6, 12, 24, 48

(1, 3)
(2, 6)
(3, 12)
(4, 24)
(5, 48)



Example 3
Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05

③ Find 15^{th} term

$a_1 = 20$ $r = 1.05$ $n = 15$

$a_n = a_1 r^{n-1}$

$a_{15} = 20(1.05)^{15-1}$

$a_{15} = 20(1.05)^{14}$

$a_{15} = 39.60$

Example 4 – *Finding a Term of a Geometric Sequence*

4.

Find the 12th term of the geometric sequence

5, 15, 45, ...

4 Find 12th term geometric 5, 15, 45

$a_1 = 5$

$r = \frac{15}{5} = 3$

$n = 12$

$a_{12} = 5(3)^{12-1}$

$a_{12} = 5(3)^{11}$

$a_{12} = 885{,}735$

Geometric Sequences

When you know *any* two terms of a geometric sequence, you can use that information to find *any other* term of the sequence.

Example 5

The fourth term of a geometric sequence is 125, and the tenth term is 125/64. Find the 14th term. (Assume that the terms of the sequence are positive.)

5) 4th term is 125

10th term $\frac{125}{64}$

Find 14th term

a_1

$a_4 = 125$

$a_4 = a_1 r^3$

a_7

$a_{10} = \frac{125}{64}$

$a_{10} = a_1 r^9$

$a_{10} = a_1 r^3 \cdot r^6$

$\frac{1}{125} \cdot \frac{125}{64} = 125 r^6 \cdot \frac{1}{125}$

$\frac{1}{64} = r^6$

$\left(\frac{1}{2}\right)^6 = r^6$

$r = \frac{1}{2}$

a_{11}

$a_{14} = \frac{125}{1024}$

$a_n = a_1 r^{n-1}$

$a_n = 125\left(\frac{1}{2}\right)^{14}$

$a_{11} = 125\left(\frac{1}{2}\right)^{10}$

$a_{11} = \frac{125}{1024}$

$a_{14} = \frac{125}{1024}$

The Sum of a Finite Geometric Sequence

The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ \ldots,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right)$.

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} \qquad S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

Example 6 – *Sum of a Finite Geometric Sequence*

6. Find the sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$.

$4(.3)^0, 4(.3)^1, 4(.3)^2, \ldots\ldots 4(.3)^{11}$

$4, 1.2, \ldots$

6 Find sum $\sum_{i=1}^{12} 4(0.3)^{i-1}$

$a_1 = 4(0.3)^{1-1} = 4(0.3)^0 = 4 \cdot 1 = 4$

$r = 0.3$

$n = 12$

$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

$S_{12} = 4\left(\frac{1-0.3^{12}}{1-0.3}\right)$

$S_{12} \approx 5.714$

The Sum of a Finite Geometric Sequence

When using the formula for the sum of a finite geometric sequence, be careful to check that the sum is of the form

$$\sum_{i=1}^{n} a_1 r^{i-1}.$$

Exponent for r is $i - 1$.

For a sum that is not of this form, you must adjust the formula. For instance, if the sum in Example 6 were $\sum_{i=1}^{12} 4(0.3)^i$, then you would evaluate the sum as follows.

$$\sum_{i=1}^{12} 4(0.3)^i = 4(0.3) + 4(0.3)^2 + 4(0.3)^3 + \cdots + 4(0.3)^{12}$$

$a_1 = 4(.3)$ $r = .3$ $n = 12$

$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

$S_{12} = 4(.3)\left(\frac{1-.3^{12}}{1-.3}\right)$

$= 1.714$

The Sum of a Finite Geometric Sequence

$= 4(0.3) + [4(0.3)](0.3) + [4(0.3)](0.3)^2 + \cdots + [4(0.3)](0.3)^{11}$

$$= 4(0.3)\left[\frac{1 - (0.3)^{12}}{1 - 0.3}\right] \qquad a_1 = 4(0.3),\ r = 0.3,\ n = 12$$

$$\approx 1.714$$

Geometric Series

Geometric Series

The summation of the terms of an infinite geometric *sequence* is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite* geometric *sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite* geometric *series*.

Specifically, if the common ratio r has the property that $|r| < 1$, it can be shown that r^n approaches zero as n increases without bound.

Think about sum formula

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \quad \text{as } n \to \infty \quad r^n \to 0$$

$$\text{So } S_n = a_1\left(\frac{1-0}{1-r}\right) = a_1\left(\frac{1}{1-r}\right) = \frac{a_1}{1-r} \qquad |r|<1$$

Geometric Series

Consequently,

$$a_1\left(\frac{1-r^n}{1-r}\right) \longrightarrow a_1\left(\frac{1-0}{1-r}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

0

The following summarizes this result.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1-r}.$$

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1-r}$$

Note that if $|r| \geq 1$, the series does not have a sum.

Example 7 – *Finding the Sum of an Infinite Geometric Series*

7.

Find each sum.

a. $\sum_{n=0}^{\infty} 4(0.6)^n$

7a $\sum_{n=0}^{\infty} 4(0.6)^n$ $a_1 = 4(0.6)^0$ $a_1 = 4$

$r = 0.6$ has a sum

$S = \frac{a_1}{1-r} = \frac{4}{1-.6} = \frac{4}{.4}$

$S = 10$

b. 3 + 0.3 + 0.03 + 0.003 + ⋯

7b $3 + 0.3 + 0.03 + 0.003 + \ldots$

$a_1 = 3$

$r = \frac{.3}{3} = \frac{3}{10} \cdot \frac{1}{3} = \frac{1}{10}$

$S = \frac{a_1}{1-r} = \frac{3}{1-.1} = \frac{3}{.9} = \frac{3}{\frac{9}{10}}$

$= 3 \cdot \frac{10}{9}$

$S = \frac{10}{3}$

Application

Example 8 – *Increasing Annuity*

8.
An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an **increasing annuity**.)

8 Remember Interest Formula

$P=50 \quad n=12 \quad r=0.03 \quad t=2$

$A=P\left(1+\frac{r}{n}\right)^{nt}$

$A_1=50\left(1+\frac{.03}{12}\right)^{12\cdot\frac{1}{12}}=50(1.0025)^1$ ← think of A_1 as 1st term

$a_1=5(1.0025)^1=50.125$

$r=1.0025$

$n=24$

⋮

$A_{23}=50\left(1+\frac{.03}{12}\right)^{23}=50(1.0025)^{23}$

A_{24}

↑ total annuity balance is sum of thes

$S_n=a_1\left(\frac{1-r^n}{1-r}\right)$

$S_n=50.125\left(\frac{1-1.0025^{24}}{1-1.0025}\right)$

$S_{24}=1238.23$

The balance in that increasing annuity after 24 months is \$1238.23.

Example 8 – *Increasing Annuity*

An investor deposits \$50 on the first day of each month in an account that pays 3% interest, compounded monthly. What is the balance at the end of 2 years? (This type of investment plan is called an **increasing annuity.**)

Solution:

To find the balance in the account after 24 months, consider each of the 24 deposits separately.
The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.03}{12}\right)^{24} = 50(1.0025)^{24}.$$

Example 8 – *Solution*

cont'd

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.03}{12}\right)^{23}$$

$$= 50(1.0025)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.03}{12}\right)^{1}$$

$$= 50(1.0025).$$

Example 8 – *Solution*

cont'd

The total balance in the annuity will be the sum of the balances of the 24 deposits.

Using the formula for the sum of a finite geometric sequence, with $A_1 = 50(1.005)$ and $r = 1.005$, and $n = 24$, you have

$$S_n = A_1\left(\frac{1 - r^n}{1 - r}\right)$$ Sum of a finite geometric sequence

$$S_{24} = 50(1.0025)\left[\frac{1 - (1.0025)^{24}}{1 - 1.0025}\right]$$ Substitute 50(1.005) for A_1, 1.005 for r, and 24 for n.

$$\approx \$1238.23.$$ Use a calculator.

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