9.2 Arithmetic Sequences and Partial Sums

Objectives

- Recognize, write, and find the *n*th terms of arithmetic sequences.
- Find *n*th partial sums of arithmetic sequences.
- Use arithmetic sequences to model and solve real-life problems.

Arithmetic Sequences

Arithmetic Sequences

A sequence whose consecutive terms have a common difference is called an **arithmetic sequence.**

Definition of Arithmetic Sequence

A sequence is **arithmetic** when the differences between consecutive terms are the same. So, the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is arithmetic when there is a number d such that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d.$$

The number d is the **common difference** of the arithmetic sequence.

Example 1 – *Examples of Arithmetic Sequences*

1. Write the first 4 terms & find the common difference.

a. The sequence whose *n*th term is $4n + 3$ is arithmetic.

1a $a_n = 4n+3$

$4\cdot1+3,\ 4\cdot2+3,\ 4\cdot3+3,\ 4\cdot4+3$

$7, 11, 15, 19$

$d = 19-15 = 15-11 = 11-7 = 4$

$d=4$

b. The sequence whose *n*th term is $7 - 5n$ is arithmetic.

1b $7-5n$

$7-5(1), 7-5(2), 7-5(3), 7-5(4)$

$2, -3, -8, -13$

$d = -3-2 = -5$

$d = -5$

Example 1 – *Examples of Arithmetic Sequences* cont'd

c. The sequence whose *n*th term is $\frac{1}{4}(n + 3)$ is arithmetic.

1c

$\frac{1}{4}(1+3), \frac{1}{4}(2+3), \frac{1}{4}(3+3), \frac{1}{4}(4+3)$

$\frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}$

$1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$

$d = \frac{5}{4} - 1$

$\frac{5}{4} - \frac{4}{4}$

$d = \frac{1}{4}$

Arithmetic Sequences

The sequence 1, 4, 9, 16, . . . , whose nth term is n^2, is *not* arithmetic. The difference between the first two terms is

$$a_2 - a_1 = 4 - 1 = 3$$

but the difference between the second and third terms is

$$a_3 - a_2 = 9 - 4 = 5.$$

The nth term of an arithmetic sequence can be derived from the following pattern.

$a_1 = a_1 + 0d$ 1st term.

$a_2 = a_1 + d$ 2nd term.

$a_3 = a_1 + d + d = a_1 + 2d$

$a_4 = a_1 + 2d + d = a_1 + 3d$

$a_5 = a_1 + 4d$

$a_n = a_1 + (n-1)d$

Arithmetic Sequences

The following definition summarizes this result.

The *n*th Term of an Arithmetic Sequence

The *n*th term of an arithmetic sequence has the form

$$a_n = a_1 + (n - 1)d$$

where d is the common difference between consecutive terms of the sequence and a_1 is the first term.

$a_n = a_1 + (n-1)d$

d = common difference

a_1 = 1^{st} term

Example 2 – *Finding the nth Term*

2.
Find a formula for the *n*th term of the arithmetic sequence whose common difference is 3 and whose first term is 2.

(2) $d = 3 \quad a_1 = 2$

$a_n = a_1 + (n-1)d$

$a_n = 2 + (n-1)3$

$a_n = 2 + 3n - 3$

$a_n = 3n - 1$

$3(1)-1, \ 3(2)-1, \ 3(3)-1, \ 3(4)-1$

$\underbrace{2, 5}_{+3}, \underbrace{\ 8}_{+3}, \underbrace{\ 11}_{+3}$

Example 2 – *Solution*

cont'd

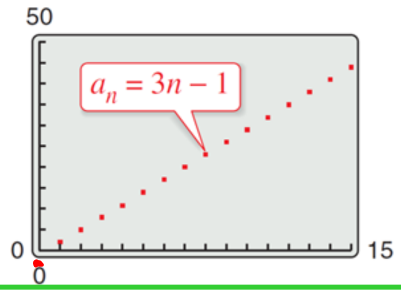
So, the formula for the *n*th term is

$$a_n = 3n - 1.$$

The sequence therefore has the following form.

$$2, 5, 8, 11, 14, \ldots, 3n - 1, \ldots.$$

The figure below shows a graph of the first 15 terms of the sequence.



Example 2 – *Solution*

cont'd

Notice that the points lie on a line. This makes sense because a_n is a linear function of n.

In other words, the terms "arithmetic" and "linear" are closely connected.

Example 3
The fourth term of an arithmetic sequence is 20, and the 13th term is 65. Write the first 11 terms of this sequence.

③ $a_4 = 20$ $a_{13} = 65$ write 1st 11 terms

1 2 3 4 5 6 7 8 9 10 11 12 13

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55 60 65 ✓

1st 10th

$a_n = a_1 + (n-1)d$
$65 = 20 + (10-1)d$
$65 = 20 + 9d$
$45 = 9d$
$5 = d$

$a_4 = 20$ → $n = 4$ $a_4 = a_n = 20$
$a_{13} = 65$

$$\begin{array}{r} 65 \\ -20 \\ \hline 45 \end{array} \qquad \begin{array}{r} 13 \\ -4 \\ \hline 9 \end{array}$$

$$\frac{65-20}{13-4} = \frac{45}{9} = 5 = d$$

$a_n = a_1 + (n-1)d$
$20 = a_1 + (4-1)5$
$20 = a_1 + 15$
$5 = a_1$

Arithmetic Sequences

$a_1 = 4$ $d = 3$ 4, 7, 10, 13,

When you know the nth term of an arithmetic sequence *and* you know the common difference of the sequence, you can find the $(n + 1)$th term by using the *recursion formula*

$$a_{n+1} = a_n + d.$$

Recursion formula

With this formula, you can find any term of an arithmetic sequence, *provided* that you know the preceding term.

For instance, when you know the first term, you can find the second term. Then, knowing the second term, you can find the third term, and so on.

Example 4
Find the ninth term of the arithmetic sequence that begins with 2 and 9.

4 Find a_9 arithmetic, begins with 2,9

2, 9

$d = 9 - 2 = 7$

Method 1 $d = 9 - 2 = 7$

$a_n = a_1 + (n-1)d$

$a_9 = 2 + (9-1)7$

$a_9 = 2 + 8 \cdot 7$

$a_9 = 58$

Method 2 $d = 9 - 2 = 7$

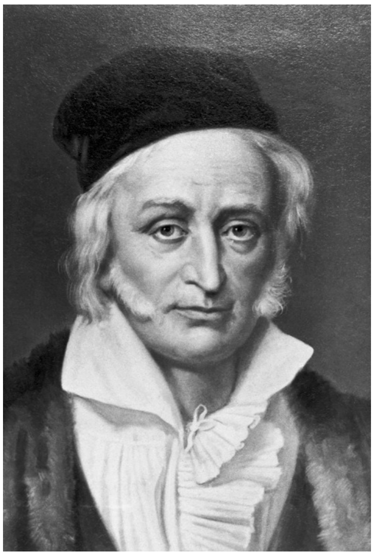
2 enter

ans+7 enter 9

enter 16

⋮

8 times 58



A teacher of Carl Friedrich Gauss (1777-1855) asked him to add all the integers from 1 to 100. When Gauss returned with the correct answer after only a few moments, the teacher could only look at him in astounded silence. Ask me to show you what Gauss did.

$\overset{a_1}{1} + 2 + 3 + \cdots + 98 + 99 + \overset{n}{100}$

$\underset{a_n}{100} + 99 + 98 + \cdots + 3 + 2 + 1$

$101 + 101 + 101 + \cdots + 101 + 101 + 101$

$\frac{100(101)}{2}$ $\quad$ $\frac{n(a_1 + a_n)}{2}$

The Sum of a Finite Arithmetic Sequence

The Sum of a Finite Arithmetic Sequence

There is a formula for the *sum* of a finite arithmetic sequence.

The Sum of a Finite Arithmetic Sequence

The sum of a finite arithmetic sequence with n terms is $S_n = \frac{n}{2}(a_1 + a_n)$.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Example 5 – *Sum of a Finite Arithmetic Sequence*

5.Find the sum: 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19.

5 Find sum: 1+3+5+7+9+11+13+15+17+19

Is it Arithmetic:

3-1=2 yes

5-3=2 d=2

S_{10}

$n=10$

$a_1=1$

$a_{10}=19$

$S_n=\frac{n}{2}(a_1+a_n)$

$S_{10}=\frac{10}{2}(1+19)$

$S_{10}=5\cdot 20$

$S_{10}=100$

Example 6

Find the sum of the integers

(a) from 1 to 100 and

$a_1 = 1 \quad a_{100} = 100$

$S_{100} = \frac{100}{2}(1+100)$

$= 50(101)$

$S_{100} = 5050$

(b) from 1 to N

$a_1 = 1 \quad a_n = N$

$n = N$

$S_N = \frac{N}{2}(1+N)$

The Sum of a Finite Arithmetic Sequence

The sum of the first n terms of an infinite sequence is the *nth partial sum.*

The nth partial sum can be found by using the formula for the sum of a finite arithmetic sequence.

Example 7
Find the 150th partial sum of the arithmetic sequence
5, 16, 27, 38, 49, ...

Arithmetic Sequences

$a_n = a_1 + (n-1)d$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d)$

Ex7 150^{th} partial sum arith. sequence
5, 16, 27, 38, 49

$n = 150$

$a_{150} = \underline{1644}$

$a_1 = 5$

$d = 16 - 5 = 11$

$a_{150} = 5 + (150-1)11$

$a_{150} = 1644$

$S_{150} = \frac{150}{2}(5 + 1644)$

$S_{150} = 123675$

Example 8

Find the 16th partial sum of the arithmetic sequence

100, 95, 90, 85, 80, ...

8 S_{16}

$a_1 = 100$

$d = 95 - 100 = -5$

$n = 16$

$a_n = a_1 + (n-1)d$

$a_{16} = 100 + (16-1)(-5)$

$a_{16} = 25$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{16} = \frac{16}{2}(100 + 25)$

$S_{16} = 1000$

Application

Example 9 – *Total Sales*

9. A small business sells $10,000 worth of skin care products during its first year. The owner of the business has set a goal of increasing annual sales by $7500 each year for 9 years. Assuming that this goal is met, find the total sales during the first 10 years this business is in operation.

9) This is arithmetic

$a_1 = 10000$

$d = 7500$

$n = 10$

$a_n = a_1 + (n-1)d$

$a_{10} = 10000 + (10-1)7500$

$a_{10} = 77500$

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_{10} = \frac{10}{2}(10000 + 77500)$

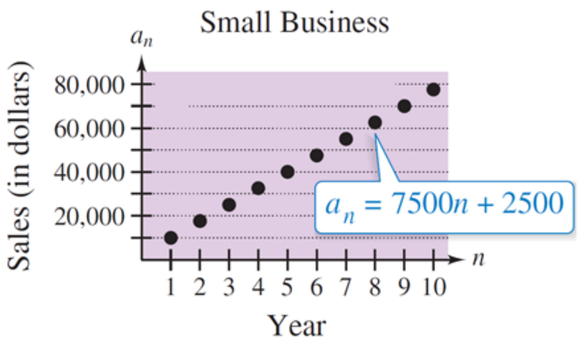
$S_{10} = 437,500$

The total amont of Sales for skin care products after the first ten years is $437,500.

Example 9 – *Solution*

cont'd

$= 77{,}500$ See figure.



The sum of the first 10 terms of the sequence is

$$S_{10} = \frac{n}{2}(a_1 + a_{10})$$ *n*th partial sum formula

$$= \frac{10}{2}(10{,}000 + 77{,}500)$$ Substitute 10 for n, 10,000 for a_1, and 77,500 for a_{10}.

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