9.1 Sequences and Series

Objectives

- Use sequence notation to write the terms of sequences.
- Use factorial notation.
- Use summation notation to write sums.
- Find the sums of series.
- Use sequences and series to model and solve real-life problems.

Sequences

Sequences

In mathematics, the word *sequence* is used in much the same way as in ordinary English.

Saying that a collection is listed in *sequence* means that it is ordered so that it has a first member, a second member, a third member, and so on.

x | 1 2 3 4 x | 1 2 3 4 ... x | 1 2 3 4

Two examples are y | 1, 2, 3, 4 . . . and 1, 3, 5, 7 . . . . y | . . . .

Mathematically, you can think of a sequence as a *function* whose domain is the set of positive integers.

Sequences

Rather than using function notation, however, sequences are usually written using subscript notation, as indicated in the following definition.

Definition of Sequence

An **infinite sequence** is a function whose domain is the set of positive integers. The function values

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

are the **terms** of the sequence. When the domain of the function consists of the first n positive integers only, the sequence is a **finite sequence.**

Sequences

On occasion it is convenient to begin subscripting a sequence with 0 instead of 1 so that the terms of the sequence become

$$a_0, a_1, a_2, a_3, \ldots .$$

When this is the case, the domain includes 0.

Example 1(a) – *Writing the Terms of a Sequence*

1a.

The first four terms of the sequence given by $a_n = 3n - 2$ are

1a. $a_n = 3n - 2$

$1, 4, 7, 10, \ldots$

$n=1 \quad a_1 = 3(1) - 2 = 1$

$n=2 \quad a_2 = 3(2) - 2 = 4$

$n=3 \quad a_3 = 3(3) - 2 = 7$

$n=4 \quad a_4 = 3(4) - 2 = 10$

Example 1(b) – *Writing the Terms of a Sequence* cont'd

1b.
The first four terms of the sequence given by
$a_n = 3 + (-1)^n$ are

1b $a_n = 3 + (-1)^n$

$n=1$ $a_1 = 3 + (-1)^1 = 3 + (-1) = 2$

$n=2$ $a_2 = 3 + (-1)^2 = 3 + 1 = 4$

$n=3$ $a_3 = 3 + (-1)^3 = 3 + (-1) = 2$

$n=4$ $a_4 = 3 + (-1)^4 = 3 + 1 = 4$

2, 4, 2, 4,

Sequence whose terms alternate signs.

Example 2 Write the first four terms of the sequence given by

$$a_n = \frac{(-1)^n}{2n + 1}$$

$-\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}$

2 $a_n = \frac{(-1)^n}{2n+1}$

$a_1 = \frac{(-1)^1}{2(1)+1} = \frac{-1}{3} = -\frac{1}{3}$

$a_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5} = \frac{1}{5}$

$a_3 = \frac{(-1)^3}{2(3)+1} = \frac{-1}{7} = -\frac{1}{7}$

$a_4 = \frac{(-1)^4}{2(4)+1} = \frac{1}{9} = \frac{1}{9}$

Sequences

Simply listing the first few terms is not sufficient to define a unique sequence—the nth term *must be given*.

To see this, consider the following sequences, both of which have the same first three terms.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \ldots, \frac{6}{(n+1)(n^2-n+6)}, \ldots$$

n	1	2	3	4	5
$2n$	2	4	6	8	10
$3n$	3	6	9	12	15
$4n$	4	8	12	16	20
$5n$	5	10	15	20	25
n^2	1	4	9	16	25
2^n	2	4	8	16	32
$(-1)^n$	-1	1	-1	1	-1
$(-1)^{n+1}$	1	-1	1	-1	1

Example 3 Write an expression for the n^{th} term

3a. 1, 3, 5, 7, ...

$n: 1, 2, 3, 4$

$2n: 2, 4, 6, 8$

$2n-1: 1\ 3\ 5\ 7$

$a_n = 2n-1$

3b. 2, -5, 10, -17, ...

$n: 1\quad 2\quad 3\quad 4$

$n^2: 1\ 4\ 9\ 16$

$n^2+1: 1\ 5\ 10\ 17$

$(-1)^{n+1}(n^2+1) = 1, -5, 10, -17$

$$\boxed{a_n = (-1)^{n+1}(n^2+1)}$$

n	1	2	3	4	5
$2n$	2	4	6	8	10
$3n$	3	6	9	12	15
$4n$	4	8	12	16	20
$5n$	5	10	15	20	25
n^2	1	4	9	16	25
2^n	2	4	8	16	32
$(-1)^n$	-1	1	-1	1	-1
$(-1)^{n+1}$	1	-1	1	-1	1

Sequences

Some sequences are defined **recursively.** To define a sequence recursively, you need to be given one or more of the first few terms.

All other terms of the sequence are then defined using previous terms.

A Recursive Sequence

Example 4 Write the first 5 terms of the sequence defined recursively as...

$a_1 = 3 \quad a_k = 2a_{k-1} + 1 \quad k \geq 2$

$3, 7, 15, 31, 63$

4 $a_1 = 3 \quad a_k = 2a_{k-1} + 1 \quad k \geq 2$

$k=2 \quad a_2 = 2a_1 + 1 = 2(3) + 1 = 7$

$k=3 \quad a_3 = 2a_{3-1} + 1 = 2a_2 + 1 = 2(7) + 1 = 15$

$k=4 \quad a_4 = 2a_3 + 1 = 2(15) + 1 = 31$

$a_5 = 2a_4 + 1 = 2(31) + 1 = 63$

The Fibonacci Sequence: A Recursive Sequence

Example 5 The fibonacci sequence is defined recursively, as follows,

$$a_0 = 1 \quad a_1 = 1 \quad a_k = a_{k-2} + a_{k-1} \quad k \geq 2$$

Write the first 6 terms of this sequence.

5 $a_0 = 1 \quad a_1 = 1 \quad a_k = a_{k-2} + a_{k-1} \quad k \geq 2$

$1, 1, 2, 3, 5, 8, \ldots$

1 $a_0 = 1$

2 $a_1 = 1$

3 $a_2 = a_{2-2} + a_{2-1} = a_0 + a_1 = 1 + 1 = 2$

4 $a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 1 + 2 = 3$

5 $a_4 = a_2 + a_3 = 2 + 3 = 5$

6 $a_5 = a_3 + a_4 = 3 + 5 = 8$

$\cancel{a_6 = a_4 + a_5 = 5 + 8 = 13}$

$3! = 1 \cdot 2 \cdot 3 = 6$!

Factorial Notation

Factorial Notation

Some very important sequences in mathematics involve terms that are defined with special types of products called **factorials.**

Definition of Factorial

If n is a positive integer, then $\boldsymbol{n}$ **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n.$$

As a special case, zero factorial is defined as $0! = 1$.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 2n$$

Factorial Notation

Notice that $0! = 1$ and $1! = 1$.

Here are some other values of $n!$.

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

Factorial Notation

Factorials follow the same conventions for order of operations as do exponents.

For instance,

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$

whereas

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 2n.$$

Example 6 – *Writing the Terms of a Sequence Involving Factorials*

6.
Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}.$$

Begin with $n = 0$.

6 $a_n = \frac{2^n}{n!}$

$1, 2, 2, \frac{4}{3}, \frac{8}{3}$

$n=0$ $a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1$

$n=1$ $a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$

$n=2 \rightarrow a_2 = \frac{2^2}{2!} = \frac{4}{2\cdot 1} = \frac{4}{2} = 2$

$n=3 \rightarrow a_3 = \frac{2^3}{3!} = \frac{8}{1\cdot 2\cdot 3} = \frac{8}{6} = \frac{4}{3}$

$n=4 \rightarrow a_4 = \frac{4^3}{4!} = \frac{4\cdot \cancel{4}^{3}\cdot \cancel{4}^{1}}{1\cdot \cancel{2}_{1}\cdot 3\cdot \cancel{4}_{1}} = \frac{8}{3}$

Factorial Notation

When working with fractions involving factorials, you will often be able to reduce the fractions to simplify the computations.

Example 7 Simplify the factorial expression.

a. $\dfrac{8!}{2!6!} = \dfrac{\cancel{8}^{4}\cdot 7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}_{1}\cdot 1\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = \dfrac{8\cdot 7\cdot \cancel{6!}}{2!\,\cancel{6!}} = \dfrac{8\cdot 7}{2\cdot 1} = 28$

$n \quad n-1 \quad (n-2)!$

b. $\dfrac{n!}{(n-1)!} = \dfrac{n\cdot\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

Σ

Summation Notation

Summation Notation

A convenient notation for the sum of the terms of a finite sequence is called **summation notation** or **sigma notation**. It involves the use of the uppercase Greek letter sigma, written as Σ.

Definition of Summation Notation

The sum of the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$$

where i is called the **index of summation,** n is the **upper limit of summation,** and 1 is the **lower limit of summation.**

Σ

upper limit

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

lower limit

index

Example 8 – *Summation Notation for a Sum*

a. $\sum_{i=1}^{5} 3i$ =

8a

$$\sum_{i=1}^{5} 3i = 3(1)+3(2)+3(3)+3(4)+3(5)$$
$$= 3+6+9+12+15$$
$$= 45$$

b. $\sum_{k=3}^{6} (1 + k^2)$ =

8b

$$\sum_{k=3}^{6} (1+k^2) = (1+3^2)+(1+4^2)+(1+5^2)+(1+6^2)$$
$$= 10+17+26+37$$
$$= 90$$

Example 8 – *Summation Notation for a Sum* cont'd

c. $\sum_{i=0}^{8} \frac{1}{i!} =$

8c $\sum_{i=0}^{8} \frac{1}{i!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$

Example 8 – *Summation Notation for a Sum* cont'd

c. $$\boxed{\sum_{i=0}^{8} \frac{1}{i!}} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40{,}320}$$

$$\approx \boxed{2.71828}$$ ⟸ Notice anything about this answer...

Example 8 – *Summation Notation for a Sum* cont'd

For this summation, note that the sum is very close to the irrational number $e \approx 2.718281828$.

It can be shown that as more terms of the sequence whose nth term is $1/n!$ are added, the sum becomes closer and closer to e.

Summation Notation

Properties of Sums

1. $\sum_{i=1}^{n} c = cn$, $\quad c$ is a constant.

2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$, $\quad c$ is a constant.

3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Series

Series

Many applications involve the sum of the terms of a finite or infinite sequence. Such a sum is called a **series.**

Definition of Series

Consider the infinite sequence $a_1, a_2, a_3, \ldots, a_i, \ldots$.

1. The sum of the first n terms of the sequence is called a **finite series** or the ***n*th partial sum** of the sequence and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

2. The sum of all the terms of the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots = \sum_{i=1}^{\infty} a_i.$$

Example 9 – *Finding the Sum of a Series*

9.

For the series $\sum_{i=1}^{\infty} \frac{3}{10^i}$

find (a) the third partial sum and (b) the sum.

9a

$$\sum_{i=1}^{3} \frac{3}{10^i} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} \quad \ldots$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \quad \ldots$$

$$= .3 + .03 + .003 \quad \ldots$$

$$= .333$$

9b

$$\sum_{i=1}^{\infty} \frac{3}{10^i} = \frac{1}{3}$$

Application

Application

Sequences have many applications in business and science. Example 10 illustrates such an application.

Example 10 – *Compound Interest*

10

10. An investor deposits \$5000 in an account that earns 3% interest compounded quarterly. The balance in the account after *n* quarters is given by

$$A_n = 5000\left(1 + \frac{0.03}{4}\right)^n, \quad n = 0, 1, 2, \ldots$$

$A_n = 5000\left(1+\frac{0.03}{4}\right)^n$ $n=0,1,3\ldots$

a. Write the first three terms of the sequence. 5000, 5037.50, 5075.28

$A_0 = 5000\left(1+\frac{0.03}{4}\right)^0 = 5000$

$A_1 = 5000\left(1+\frac{0.03}{4}\right)^1 = 5037.50$

$A_2 = 5000\left(1+\frac{0.03}{4}\right)^2 = 5075.28$

b. Find the balance in the account after 10 years by computing the 40th term of the sequence.

A_{40} \$ 6741.74

9.1 Sequences and Series

Objectives

- Use sequence notation to write the terms of sequences.
- Use factorial notation. !
- Use summation notation to write sums. Σ
- Find the sums of series.
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