You can use the Law of Cosines to solve an oblique triangle for the remaining two cases: when you are given the measures of three sides (SSS) or the measures of two sides and their included angle (SAS).

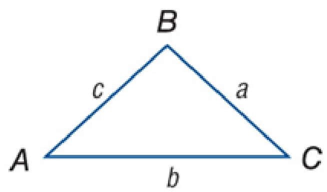
Key Concept Law of Cosines

In △ABC, if sides with lengths a, b, and c are opposite angles with measures A, B, and C, respectively, then the following are true.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c = \sqrt{a^2 + b^2 - 2ab\cos C}$$

StudyTip

Check for Reasonableness Because a triangle can have at most one obtuse angle, it is wise to find the measure of the largest angle in a triangle first, which will be the angle opposite the longest side. If the largest angle is obtuse, then you know that the other two angles must be acute. If the largest angle is acute, the remaining two angles must still be acute.

StudyTip

Law of Cosines Notice that the angle referenced in each equation of the Law of Cosines corresponds to the side length on the other side of the equation.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\frac{c^2 - a^2 - b^2}{-2ab} = \frac{-\cancel{2ab}\cos C}{-\cancel{2ab}}$$

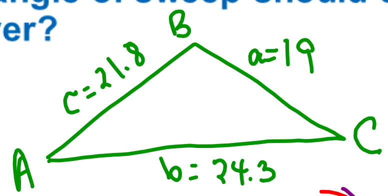
$$\cos C = \frac{c^2 - a^2 - b^2}{-2ab}$$

$$C = \cos^{-1}\left(\frac{(c^2 - a^2 - b^2)}{(-2ab)}\right) \qquad C = \cos^{-1}\left(\frac{(a^2 + b^2 - c^2)}{(2ab)}\right)$$

Real-World EXAMPLE 5 Apply the Law of Cosines (SSS)

LANDSCAPING A triangular area of lawn has a sprinkler located at each vertex. If the sides of the lawn are $a = 19$ feet, $b = 24.3$ feet, and $c = 21.8$ feet, what angle of sweep should each sprinkler be set to cover?

5



$$C = \cos^{-1}\left(\frac{c^2 - a^2 - b^2}{-2ab}\right)$$

$$B = \cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right)$$

$$B = \cos^{-1}\left(\frac{(24.3)^2 - 19^2 - 21.8^2}{(-2 \cdot 19 \cdot 21.8)}\right)$$

$$B \approx 72.7^\circ$$

$$C = \cos^{-1}\left(\frac{(21.8^2 - 19^2 - 24.3^2)}{(-2 \cdot 19 \cdot 24.3)}\right)$$

$$C \approx 59.0^\circ$$

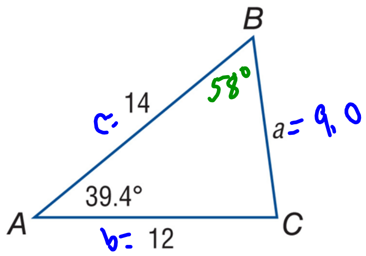
$$A = \cos^{-1}\left(\frac{(19^2 - 21.8^2 - 24.3^2)}{(-2 \cdot 21.8 \cdot 24.3)}\right)$$

$$A \approx 48.3^\circ$$

$$\boxed{A \approx 48^\circ \quad B \approx 73^\circ \quad C \approx 59^\circ}$$

EXAMPLE 6 Apply the Law of Cosines (SAS)

Solve ΔABC. Round side lengths to the nearest tenth and angle measures to the nearest degree.



$a^2 = b^2 + c^2 - 2bc\cos A$

$a = \sqrt{b^2 + c^2 - 2bc\cos A}$

$a = \sqrt{12^2 + 14^2 - 2\cdot 12\cdot 14\cos 39.4^\circ}$

$a \approx \underline{9.0}$

$\frac{\sin 39.4^\circ}{9.0} = \frac{\sin B}{12}$

$9\sin B = 12\sin 39.4^\circ$

$\sin B = \frac{12\sin 39.4^\circ}{9}$

$B = \sin^{-1}\left(\frac{12\sin 39.4^\circ}{9}\right)$

$B \approx \underline{58^\circ}$

$C = 180 - (39.4 + 58)$

$C \approx 82.38^\circ$

$C \approx \underline{82^\circ}$

$\boxed{a \approx 9.0 \quad B \approx 58^\circ \quad C \approx 82^\circ}$

2 Find Areas of Oblique Triangles

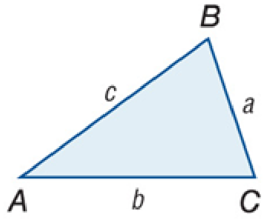
When the measures of all three sides of a triangle are known, the Law of Cosines can be used to prove **Heron's Formula** for the area of the triangle.

Key Concept Heron's Formula

If the measures of the sides of $\triangle ABC$ are a, b, and c, then the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$.



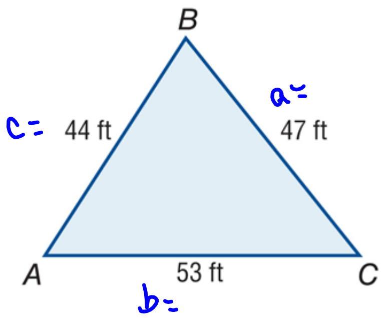
$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$

StudyTip

Semiperimeter The measure s used in Heron's Formula is called the *semiperimeter* of the triangle.

EXAMPLE 7 Heron's Formula

Find the area of ΔABC to the nearest tenth.



7. $s = \frac{47+53+44}{2}$

$s = \frac{144}{2}$

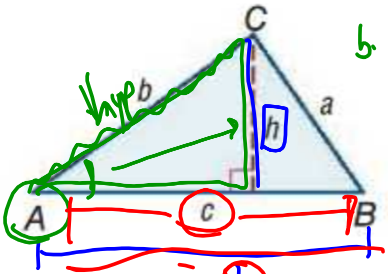
$s = 72$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$A = \sqrt{72(72-47)(72-53)(72-44)}$

$A = \sqrt{72 \cdot 28 \cdot 25 \cdot 19}$

$A \approx \underline{978.6 \text{ ft}^2}$



$b \cdot \sin A = \frac{h}{b} \cdot b$

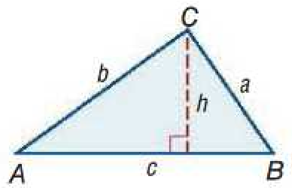
$b \sin A = h$

$A = \frac{1}{2} bh$

$A = \frac{1}{2} c b \sin A$

$A = \frac{1}{2} bc \sin A$

In the ambiguous case of the Law of Sines, you compared the length of a to the value $h = b \sin A$. In the triangle shown, h represents the length of the altitude to side c in $\triangle ABC$. You can use this expression for the height of the triangle to develop a formula for the area of the triangle.



$$\text{Area} = \frac{1}{2}ch \qquad \text{Formula for area of a triangle}$$

$$= \frac{1}{2}c(b \sin A) \qquad \text{Replace } h \text{ with } b \sin A.$$

$$= \frac{1}{2}bc \sin A \qquad \text{Simplify.}$$

By a similar argument, you can develop the formulas

$$\text{Area} = \frac{1}{2}ab \sin C \qquad \text{and} \qquad \text{Area} = \frac{1}{2}ac \sin B.$$

Notice that in each of these formulas, the information needed to find the area of the triangle is the measures of two sides and the included angle.

Key Concept — Area of a Triangle Given SAS

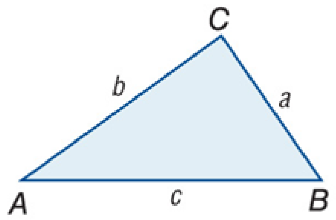
Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

Symbols

Area $= \frac{1}{2}bc \sin A$

Area $= \frac{1}{2}ac \sin B$

$\boxed{\text{Area} = \frac{1}{2}ab \sin C}$



Because the area of a triangle is constant, the formulas above can be written as one formula.

$$\text{Area} = \frac{1}{2}bc \sin A = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B$$

If the included angle measures 90°, notice that each formula simplifies to the formula for the area of a right triangle, $\frac{1}{2}$(base)(height), because $\sin 90° = 1$.

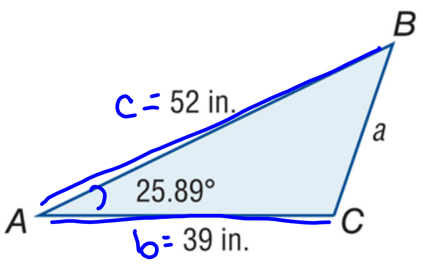
StudyTip

Area of an Obtuse Triangle This formula works for any type of triangle, including obtuse triangles. You will prove this in Lesson 5-3.

EXAMPLE 8 Find the Area of a Triangle Given SAS

Find the area of ΔABC to the nearest tenth.

8



$\text{Area} = \frac{1}{2} bc \sin A$

$\text{Area} = \frac{1}{2}(39)(52)\sin 25.89°$

$\text{Area} \approx \underline{442.8 \text{ in}^2}$

EXAMPLE 1 Guided Practice

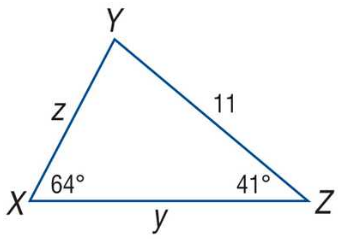
Solve ΔXYZ. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $y \approx 11.8$, $z \approx 8.0$, $Y = 75°$

B. $y \approx 8.0$, $z \approx 5.3$, $Y = 85°$

C. $y \approx 28.7$, $z \approx 18.9$, $Y = 95°$

D. $y \approx 14.6$, $z \approx 9.9$, $Y = 75°$



Real-World EXAMPLE 2 Guided Practice

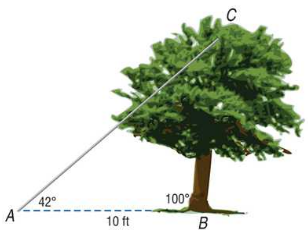
TREE A tree is leaning 10° past vertical as shown in the figure. A wire that makes a 42° angle with the ground 10 feet from the base of the tree is attached to the top of the tree. How tall is the tree?

A. 6.8 ft

B. 7.8 ft

C. 14.3 ft

D. 10.9 ft



EXAMPLE 3 Guided Practice

Find all solutions for ΔABC where $a = 18$, $b = 13$, and $A = 126°$, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $B \approx 42°$, $C \approx 12°$, $c \approx 4.6$

B. $B \approx 28°$, $C \approx 26°$, $c \approx 9.8$

C. $B \approx 36°$, $C \approx 18°$, $c \approx 6.9$

D. no solution

EXAMPLE 4 Guided Practice

Find two triangles for which $A = 24°$, $a = 13$, and $b = 15$. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $B \approx 128°$, $C \approx 28°$, $c \approx 15.0$,
$B \approx 4°$, $C \approx 152°$, $c \approx 17.3$

B. $B \approx 28°$, $C \approx 128°$, $c \approx 25.2$,
$B \approx 152°$, $C \approx 4°$, $c \approx 2.2$

C. $B \approx 28°$, $C \approx 128°$, $c \approx 25.2$,
$B \approx 62°$, $C \approx 92°$, $c \approx 31.9$

D. $B \approx 21°$, $C \approx 135°$, $c \approx 22.6$,
$B \approx 69°$, $C \approx 87°$, $c \approx 31.9$

Real-World EXAMPLE 5 Guided Practice

LOT A triangular lot has sides of 120 feet, 186 feet, and 147 feet. Find the angle across from the shortest side.

A. 9°

B. 40°

C. 49°

D. 52°

EXAMPLE 6 **Guided Practice**

Solve ΔMNP if $M = 54°$, $n = 17$, and $p = 12$. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $m \approx 193.2$, $N \approx 4°$, $P \approx 122°$

B. $m \approx 13.9$, $N \approx 82°$, $P \approx 44°$

C. $m \approx 17.7$, $N \approx 51°$, $P \approx 75°$

D. $m \approx 16.1$, $N \approx 59°$, $P \approx 67°$

EXAMPLE 7 Guided Practice

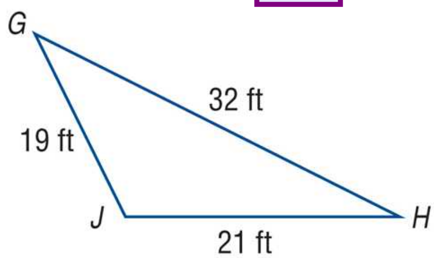
Find the area of ΔGHJ.

A. 2790.1 ft^2

B. 678.0 ft^2

C. 191.6 ft^2

D. 31.9 ft^2



EXAMPLE 8 Guided Practice

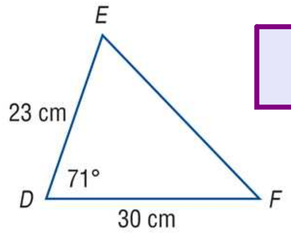
Find the area of Δ*DEF* to the nearest tenth.

A. 652.4 cm^2

B. 326.2 cm^2

C. 224.6 cm^2

D. 112.3 cm^2



EXAMPLE 1 **Guided Practice**

Solve ΔXYZ. Round side lengths to the nearest tenth and angle measures to the nearest degree.

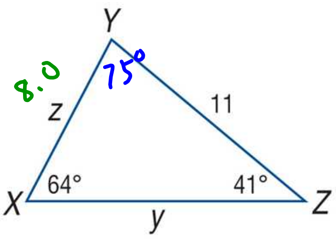
$\boxed{A}$

A. $y \approx 11.8$, $z \approx 8.0$, $Y = 75°$

B. $y \approx 8.0$, $z \approx 5.3$, $Y = 85°$

C. $y \approx 28.7$, $z \approx 18.9$, $Y = 95°$

D. $y \approx 14.6$, $z \approx 9.9$, $Y = 75°$



$Y = 180 - (64 + 41)$

$Y = 75°$

$\dfrac{\sin 64°}{11} = \dfrac{\sin 41°}{z}$

$z = \dfrac{11 \sin 41°}{\sin 64°}$

$z \approx 8.0$

$\dfrac{\sin 64°}{11} = \dfrac{\sin 75°}{y}$

$y = \dfrac{11 \sin 75°}{\sin 64°} \approx 11.8$

Real-World EXAMPLE 2 **Guided Practice**

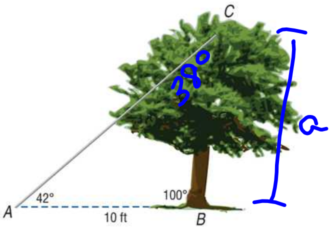
TREE A tree is leaning 10° past vertical as shown in the figure. A wire that makes a 42° angle with the ground 10 feet from the base of the tree is attached to the top of the tree. How tall is the tree?

A. 6.8 ft

B. 7.8 ft

C. 14.3 ft

D. 10.9 ft



$\boxed{D}$

$C = 180 - (42 + 100)$

$C = 38°$

$$\frac{\sin 42°}{a} = \frac{\sin 38°}{10}$$

$$a = \frac{10 \sin 42°}{\sin 38°}$$

$a \approx 10.9$ ft

EXAMPLE 3 Guided Practice

Find all solutions for ΔABC where $a = 18$, $b = 13$, and $A = 126°$, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $B \approx 42°$, $C \approx 12°$, $c \approx 4.6$

B. $B \approx 28°$, $C \approx 26°$, $c \approx 9.8$

C. $B \approx 36°$, $C \approx 18°$, $c \approx 6.9$

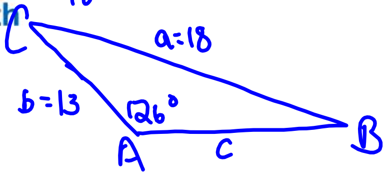
D. no solution

$\boxed{C}$

$A = 126°$ obtuse

$a > b$ so 1 Δ

$18 > 13$



$$\frac{\sin 126°}{18} = \frac{\sin B}{13}$$

$$\sin B = \frac{13 \sin 126°}{18}$$

$$B = \sin^{-1}\left(\frac{13 \sin 126°}{18}\right)$$

$$B \approx 36°$$

$$C = 180 - (126 + 36)$$

$$C \approx 18°$$

$$\frac{\sin 126}{18} = \frac{\sin 18°}{c}$$

$$c = \frac{18 \sin 18°}{\sin 126°} \approx 6.9$$

EXAMPLE 4 **Guided Practice**

Find two triangles for which $A = 24°$, $a = 13$, and $b = 15$. Round side lengths to the nearest tenth and angle measures to the nearest degree.

A. $B \approx 128°$, $C \approx 28°$, $c \approx 15.0$, $B \approx 4°$, $C \approx 152°$, $c \approx 17.3$

B. $B \approx 28°$, $C \approx 128°$, $c \approx 25.2$, $B \approx 152°$, $C \approx 4°$, $c \approx 2.2$

C. $B \approx 28°$, $C \approx 128°$, $c \approx 25.2$, $B \approx 62°$, $C \approx 92°$, $c \approx 31.9$

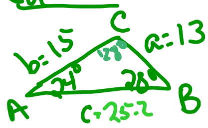
D. $B \approx 21°$, $C \approx 135°$, $c \approx 22.6$, $B \approx 69°$, $C \approx 87°$, $c \approx 31.9$

$\boxed{B}$

here is why 2 Δs (you have to be able to decide if 1, 2, or 0 Δs)

A is acute
$a < b$
$13 < 15$
find h
$\sin 24 = \frac{h}{15}$
$15 \sin 24 = h$
$h \approx 6.1$
$h < a$
$6.1 < 13$ so 2 Δs

Case 1



$\frac{\sin 24°}{13} = \frac{\sin B}{15}$

$B = \sin^{-1}\left(\frac{15 \sin 24°}{13}\right)$

$B \approx 28°$

$C = 180 - (24 + 28)$

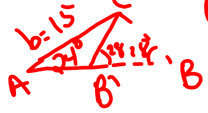
$C \approx 128°$

$\frac{\sin 24°}{13} = \frac{\sin 128°}{c}$

$c = \frac{13 \sin 128°}{\sin 24°}$

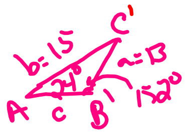
$c \approx 25.2$

Case



$B' = 180 - 28$

$B' = 152$



$C' = 180 - (152 + 24)$

$C' = 4°$

$\frac{\sin 24°}{13} = \frac{\sin 4°}{c}$

$c = \frac{13 \sin 4°}{\sin 24°}$

$c \approx 2.2$

Real-World EXAMPLE 5 **Guided Practice**

LOT A triangular lot has sides of 120 feet, 186 feet, and 147 feet. Find the angle across from the shortest side.

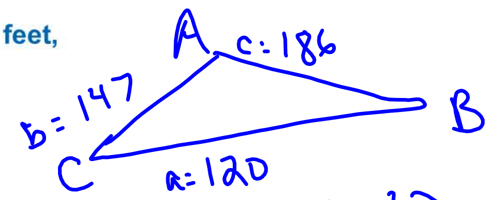
A. 9°

B. 40°

C. 49°

D. 52°

$\boxed{B}$



$$A = \cos\left(\frac{a^2 - b^2 - c^2}{-2bc}\right)$$

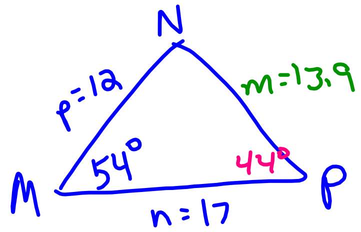
$$A = \cos\left(\frac{(120^2 - 147^2 - 186^2)}{(-2 \cdot 147 \cdot 186)}\right)$$

$$A \approx 40^\circ$$

EXAMPLE 6 Guided Practice

Solve ΔMNP if $M = 54°$, $n = 17$, and $p = 12$. Round side lengths to the nearest tenth and angle measures to the nearest degree.

$\boxed{B}$



A. $m \approx 193.2$, $N \approx 4°$, $P \approx 122°$

B. $m \approx 13.9$, $N \approx 82°$, $P \approx 44°$

C. $m \approx 17.7$, $N \approx 51°$, $P \approx 75°$

D. $m \approx 16.1$, $N \approx 59°$, $P \approx 67°$

$m = \sqrt{17^2 + 12^2 - 2 \cdot 17 \cdot 12 \cos 54°}$

$m \approx 13.9$

hint: if staying with Law of Cos find big < next (<N)

if changing to Law of Sines find little < next (<P)

I'm going to do this

$\frac{\sin 54°}{13.9} = \frac{\sin P}{12}$

$P = \sin^{-1}\left(\frac{12 \sin 54°}{13.9}\right)$

$P \approx 44°$

$N = 180 - (54 + 44)$

$N \approx 82°$

EXAMPLE 7 Guided Practice

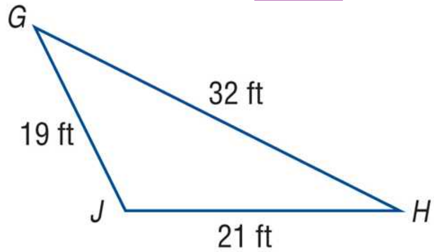
Find the area of ΔGHJ.

A. 2790.1 ft^2

B. 678.0 ft^2

C. 191.6 ft^2

D. 31.9 ft^2



$\boxed{C}$

$s = \frac{1}{2}(a+b+c)$

$s = \frac{1}{2}(19+21+32)$

$s = 36$

$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

$\text{Area} = \sqrt{36(36-19)(36-21)(36-32)}$

$\text{Area} \approx 191.6 \text{ ft}^2$

EXAMPLE 8 Guided Practice

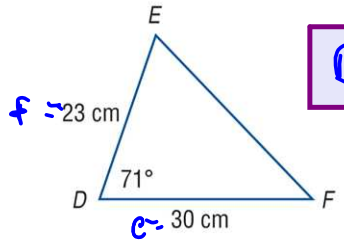
Find the area of ΔDEF to the nearest tenth.

A. 652.4 cm^2

B. 326.2 cm^2

C. 224.6 cm^2

D. 112.3 cm^2



$\boxed{B}$

$\text{Area} = \frac{1}{2}ab\sin C$

$\text{Area} = \frac{1}{2}ef\sin D$

$\text{Area} = \frac{1}{2}\cdot 30\cdot 23\sin 71°$

$\text{Area} \approx 326.2\,cm^2$

4-7 The Law of Sines & the Law of Cosines

EVEN PAGE

TOC

EQ: Can you solve oblique triangles using the Law of Sines and the Law of Cosines and find areas of oblique triangles?

How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...