2 Compositions of Trigonometric Functions

In Lesson 1-7, you learned that if x is in the domain of $f(x)$ and $f^{-1}(x)$, then

$$f[f^{-1}(x)] = x \qquad \text{and} \qquad f^{-1}[f(x)] = x.$$

Because the domains of the trigonometric functions are restricted to obtain the inverse trigonometric functions, the properties do not apply for all values of x.

For example, while $\sin x$ is defined for all x, the domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore, $\sin(\sin^{-1} x) = x$ is only true when $-1 \le x \le 1$. A different restriction applies for the composition $\sin^{-1}(\sin x)$. Because the domain of $\sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}(\sin x) = x$ is only true when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$x = \frac{\pi}{6}$

$\sin^{-1}\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(\frac{1}{2}\right)$

$\frac{\pi}{6}$

$x = \frac{5\pi}{6}$

$\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(\frac{1}{2}\right)$

$\frac{\pi}{6}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

S A
T C

ref ∠ $\frac{\pi}{6}$

These domain restrictions are summarized below.

Key Concept **Domain of Compositions of Trigonometric Functions**

$f[f^{-1}(x)] = x$	$f^{-1}[f(x)] = x$
If $-1 \leq x \leq 1$, then $\sin(\sin^{-1} x) = x$.	If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$.
If $-1 \leq x \leq 1$, then $\cos(\cos^{-1} x) = x$.	If $0 \leq x \leq \pi$, then $\cos^{-1}(\cos x) = x$.
If $-\infty < x < \infty$, then $\tan(\tan^{-1} x) = x$.	If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$.

WatchOut!

Compositions and Inverses When computing $f^{-1}[f(x)]$ with trigonometric functions, the domain appears to be $(-\infty, \infty)$. However, because the ranges of the inverse functions are restricted, coterminal angles must sometimes be found.

EXAMPLE 6 Use Inverse Trigonometric Properties

A. Find the exact value of $\sin\left(\arcsin\frac{1}{2}\right)$**, if it exists.**

6A $\sin\left(\arcsin\frac{1}{2}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
sinθ	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cosθ	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
tanθ	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 6 Use Inverse Trigonometric Properties

B. Find the exact value of $\cos^{-1}\left(\cos \frac{5\pi}{2}\right)$**, if it exists.**

6b. $\cos^{-1}\left(\cos \frac{5\pi}{2}\right) = \frac{\pi}{2}$

$\cos^{-1}(0) = \frac{\pi}{2}$

$\cos \frac{5\pi}{2} = 0$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

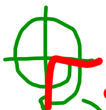
EXAMPLE 6 Use Inverse Trigonometric Properties

C. Find the exact value of $\arctan\left[\tan\left(-\frac{5\pi}{2}\right)\right]$**, if it exists.**

6c. $\arctan\left[\tan\left(-\frac{5\pi}{2}\right)\right]$

arctan (undef)

undef.



ref < $\frac{\pi}{2}$

$(0, -1)$

$(\cos\theta, \sin\theta)$

$\tan\left(-\frac{5\pi}{2}\right)$: undef

< rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
< deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

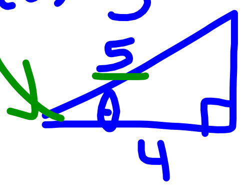
You can also evaluate the composition of two different inverse trigonometric functions.

EXAMPLE 7 Evaluate Compositions of Trigonometric Functions

Find the exact value of $\sin\left(\cos^{-1}\frac{4}{5}\right)$.

7. $\sin\left(\cos^{-1}\frac{4}{5}\right) = \frac{3}{5}$

$\cos^{-1}\frac{4}{5}$



$\sqrt{5^2-4^2} = 3$

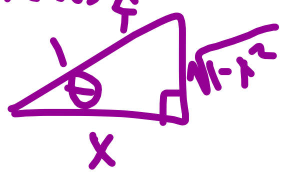
Sometimes the composition of two trigonometric functions reduces to an algebraic expression that does not involve *any* trigonometric expressions.

EXAMPLE 8 Evaluate Compositions of Trigonometric Functions

Write cot (arccos *x*) as an algebraic expression of *x* that does not involve trigonometric functions.

8 $\cot(\arccos x) = \frac{x}{\sqrt{1-x^2}} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = \frac{x\sqrt{1-x^2}}{1-x^2}$

arccos x



$\cot\theta = \frac{adj}{opp}$

StudyTip

Decomposing Algebraic Functions The technique used to convert a trigonometric expression into an algebraic expression can be reversed. Decomposing an algebraic function as the composition of two trigonometric functions is a technique used frequently in calculus.

EXAMPLE 1 **Guided Practice**

Find the exact value of $\sin^{-1} 0$.

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

EXAMPLE 2 Guided Practice

Find the exact value of $\cos^{-1}(-1)$.

A. $-\frac{\pi}{2}$

B. $\frac{3\pi}{4}$

C. π

D. $\frac{3\pi}{2}$

EXAMPLE 3 Guided Practice

Find the exact value of $\arctan\left(-\sqrt{3}\right)$.

A. $\frac{2\pi}{3}$

B. $\frac{5\pi}{6}$

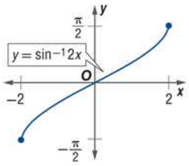
C. $-\frac{\pi}{6}$

D. $-\frac{\pi}{3}$

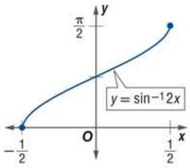
EXAMPLE 4 **Guided Practice**

Sketch the graph of $y = \sin^{-1} 2x$.

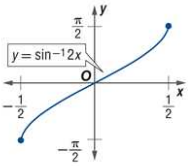
A.



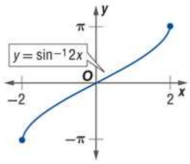
C.



B.



D.



Real-World EXAMPLE 5 **Guided Practice**

MATH COMPETITION **In a classroom, a 4 foot tall screen is located 6 feet above the floor. Write a function modeling the viewing angle θ for a student in the classroom whose eye-level when sitting is 3 feet above the floor.**

A. $\theta = \tan^{-1}\frac{10}{d} - \tan^{-1}\frac{6}{d}$

B. $\theta = \tan\frac{d}{7} - \tan\frac{d}{3}$

C. $\theta = \tan\frac{d}{10} - \tan\frac{d}{6}$

D. $\theta = \tan^{-1}\frac{7}{d} - \tan^{-1}\frac{3}{d}$

EXAMPLE 6 Guided Practice

Find the exact value of $\arcsin\left(\sin\frac{7\pi}{6}\right)$.

A. $-\frac{\pi}{6}$

B. $\frac{7\pi}{6}$

C. $\frac{11\pi}{6}$

D. $-\frac{\sqrt{3}}{2}$

EXAMPLE 7 **Guided Practice**

Find the exact value of $\cos\left(\arcsin\frac{8}{17}\right)$.

A. $\frac{15}{8}$

B. $\frac{8}{15}$

C. $\frac{17}{8}$

D. $\frac{15}{17}$

EXAMPLE 8 Guided Practice

Write cos(arctan x) as an algebraic expression of x that does not involve trigonometric functions.

A. $\frac{1}{x}$

B. $\frac{x\sqrt{x^2+1}}{x^2+1}$

C. $\frac{\sqrt{x^2+1}}{x^2+1}$

D. $\sqrt{x^2+1}$

EXAMPLE 1 Guided Practice

Find the exact value of $\sin^{-1} 0$.

A. 0

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. π

$\boxed{A}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 2 **Guided Practice**

Find the exact value of $\cos^{-1}(-1)$.

A. $-\frac{\pi}{2}$

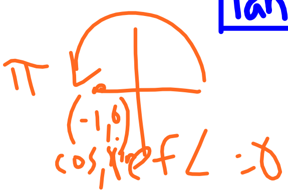
B. $\frac{3\pi}{4}$

C. π

D. $\frac{3\pi}{2}$

$\boxed{C}$

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef



EXAMPLE 3 **Guided Practice**

Find the exact value of $\arctan\left(-\sqrt{3}\right)$.

A. $\frac{2\pi}{3}$

B. $\frac{5\pi}{6}$

C. $-\frac{\pi}{6}$

D. $-\frac{\pi}{3}$

$\boxed{D}$

$\angle$ rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\angle$ deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

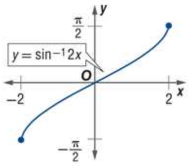
Ref$\angle$ $\frac{\pi}{3}$

$\angle$ is $-\frac{\pi}{3}$

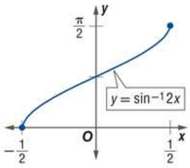
EXAMPLE 4 **Guided Practice**

Sketch the graph of $y = \sin^{-1} 2x$.

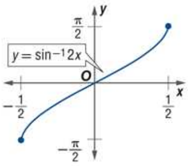
A.



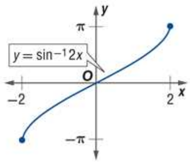
C.



B.



D.



$\boxed{B}$

$y = \sin^{-1} 2x$

$\sin y = 2x$

$\frac{1}{2}\sin y = x$

x	y
$-\frac{1}{2}$	$-\frac{\pi}{2}$
0	0
$\frac{1}{2}$	$\frac{\pi}{2}$

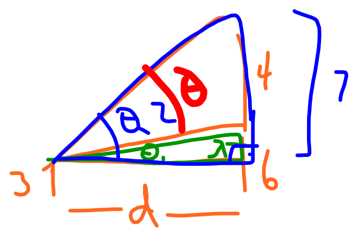
Real-World EXAMPLE 5 **Guided Practice**

MATH COMPETITION In a classroom, a 4 foot tall screen is located 6 feet above the floor. Write a function modeling the viewing angle θ for a student in the classroom whose eye-level when sitting is 3 feet above the floor.

A. $\theta = \tan^{-1}\frac{10}{d} - \tan^{-1}\frac{6}{d}$ **C.** $\theta = \tan\frac{d}{10} - \tan\frac{d}{6}$

B. $\theta = \tan\frac{d}{7} - \tan\frac{d}{3}$ **D.** $\theta = \tan^{-1}\frac{7}{d} - \tan^{-1}\frac{3}{d}$

$\boxed{D}$



$\tan\theta_2 = \frac{7}{d}$ $\tan\theta_1 = \frac{3}{d}$

$\theta_2 = \tan^{-1}\left(\frac{7}{d}\right)$ $\theta_1 = \tan^{-1}\left(\frac{3}{d}\right)$

$\theta = \theta_2 - \theta_1$

$= \tan^{-1}\left(\frac{7}{d}\right) - \tan^{-1}\left(\frac{3}{d}\right)$

EXAMPLE 6 **Guided Practice**

Find the exact value of $\arcsin\left(\sin\frac{7\pi}{6}\right)$.

A. $-\frac{\pi}{6}$

B. $\frac{7\pi}{6}$

C. $\frac{11\pi}{6}$

D. $-\frac{\sqrt{3}}{2}$

$\boxed{A}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

$\arcsin\left(\underline{\sin\frac{7\pi}{6}}\right)$

$\sin\frac{7\pi}{6}$

$-\sin\frac{\pi}{6}$

$-\frac{1}{2}$

S A
C

ref ∠ $\frac{\pi}{6}$

$\arcsin\left(-\frac{1}{2}\right)$ ref ∠ $\frac{\pi}{6}$

A

$\pi/6$ ∠ is $-\frac{\pi}{6}$

EXAMPLE 7 **Guided Practice**

Find the exact value of $\cos\left(\arcsin \frac{8}{17}\right)$.

A. $\frac{15}{8}$

B. $\frac{8}{15}$

C. $\frac{17}{8}$

D. $\frac{15}{17}$

$\boxed{D}$

$\cos\left(\arcsin \frac{8}{17}\right)$

$\cos(\theta)$

$\frac{15}{17}$

$\sin\theta = \frac{8}{17}$

EXAMPLE 8 **Guided Practice**

Write cos(arctan x) as an algebraic expression of x that does not involve trigonometric functions.

A. $\frac{1}{x}$

B. $\frac{x\sqrt{x^2+1}}{x^2+1}$

C. $\frac{\sqrt{x^2+1}}{x^2+1}$

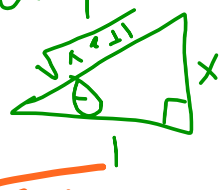
D. $\sqrt{x^2+1}$

$\boxed{C}$

$\cos(\arctan x)$

arctan x

$\tan\theta = \frac{x}{1}$



$\cos(\theta)$

$\frac{1}{\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$

$\frac{\sqrt{x^2+1}}{x^2+1}$

4-6 Inverse Trigonometric Functions

EVEN PAGE

TOC

EQ: Can you evaluate and graph inverse trigonometric functions and find compositions of trigonometric functions?

How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...