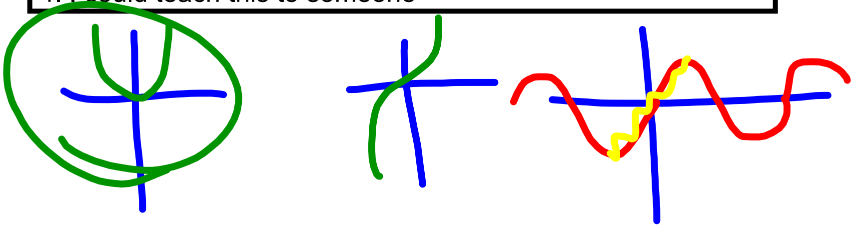
4-6 Inverse Trigonometric Functions

EVEN PAGE

TOC

EQ: Can you evaluate and graph inverse trigonometric functions and find compositions of trigonometric functions?

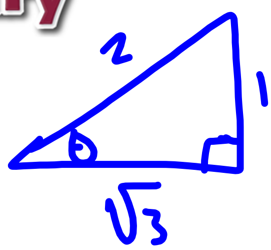
How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone



Summary: At least 3 sentences...

New Vocabulary

- arcsine function
- arccosine function
- arctangent function



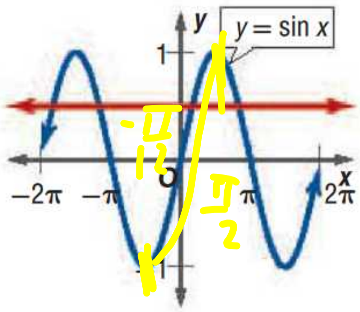
$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

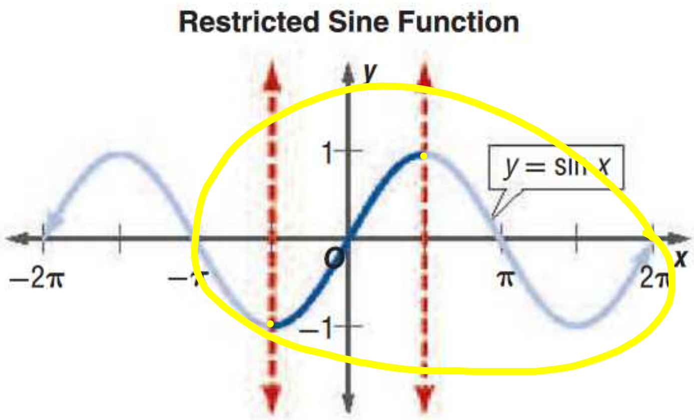
$\theta = \underline{30^\circ}$

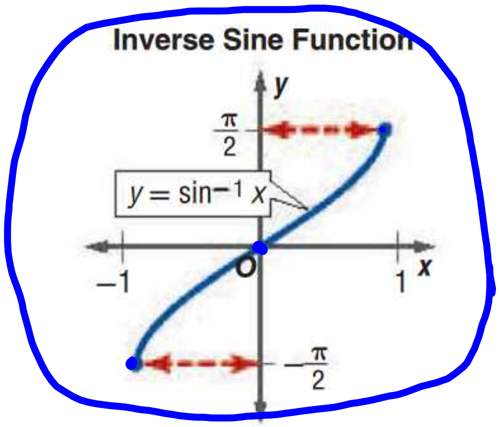
1 Inverse Trigonometric Functions

In Lesson 1-7, you learned that a function has an inverse function if and only if it is *one-to-one*, meaning that each y-value of the function can be matched with no more than one x-value. Because the sine function fails the horizontal line test, it is not one-to-one.



If, however, we restrict the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the restricted function *is* one-to-one and takes on all possible range values $[-1, 1]$ of the unrestricted function. It is on this restricted domain that $y = \sin x$ has an inverse function called the *inverse sine function* $y = \sin^{-1} x$. The graph of $y = \sin^{-1} x$ is found by reflecting the graph of the restricted sine function in the line $y = x$.





Notice that the domain of $y = \sin^{-1} x$ is $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because angles and arcs given on the unit circle have equivalent radian measures, the inverse sine function is sometimes referred to as the **arcsine function** $y = \arcsin x$.

$\frac{A}{C}$

In Lesson 4-1, you used the inverse relationship between the sine and inverse sine functions to find acute angle measures. From the graphs above, you can see that in general,

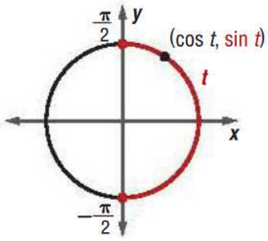
$$y = \sin^{-1} x \text{ or } y = \arcsin x \text{ iff } \sin y = x, \text{ when } -1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

iff means if and only if.

This means that $\sin^{-1} x$ or $\arcsin x$ can be interpreted as *the angle (or arc) between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *with a sine of* x. For example, $\sin^{-1} 0.5$ is the angle with a sine of 0.5.

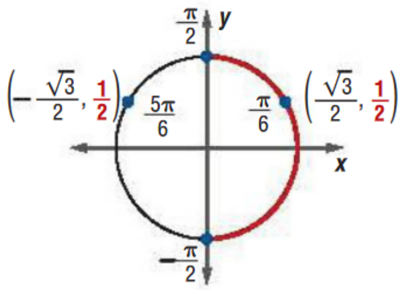
Recall that $\sin t$ is the y-coordinate of the point on the unit circle corresponding to the angle or arc length t. Because the range of the inverse sine function is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the possible angle measures of the inverse sine function are located on the right half of the unit circle, as shown.

Inverse Sine Values



You can use the unit circle to find the exact value of some expressions involving $\sin^{-1} x$ or $\arcsin x$.

Notice in Example 1a that while $\sin \frac{5\pi}{6}$ is also $\frac{1}{2}$, $\frac{5\pi}{6}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, $\sin^{-1} \frac{1}{2} \neq \frac{5\pi}{6}$.



TechnologyTip

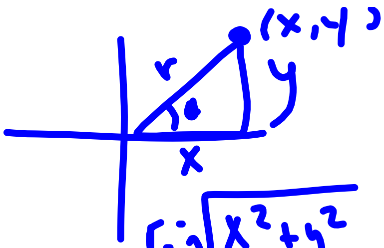
Evaluate $\sin^{-1}$ You can also use a graphing calculator to find the angle that has a sine of $\frac{1}{2}$.

```
sin⁻¹(0.5)
      .5235987756
π/6
      .5235987756
```

Make sure you select RADIAN on the MODE feature of your graphing calculator.

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

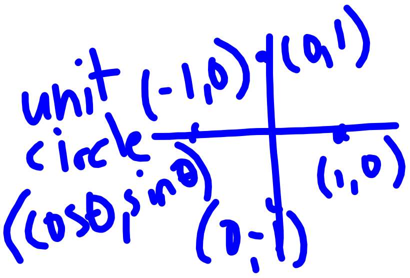
S	A
T	C

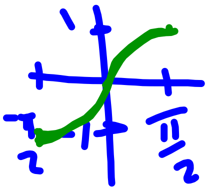


$r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

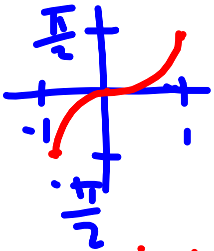




$y = \sin x$

$D: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$R: [-1, 1]$



$y = \arcsin x$

$y = \sin^{-1}(x)$

$D: [-1, 1]$

$R: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\angle$rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\angle$deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

$\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

$-\frac{\pi}{4}$

$\arcsin(3)$ not defined

EXAMPLE 1 Evaluate Inverse Sine Functions

A. Find the exact value of $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$, if it exists.

1a $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$\frac{\pi}{4}$

A

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 1 Evaluate Inverse Sine Functions

B. Find the exact value of $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$, if it exists.

1b. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

ref ∠ $\frac{\pi}{3}$

EXAMPLE 1 Evaluate Inverse Sine Functions

C. Find the exact value of $\sin^{-1}(-2\pi)$, if it exists.

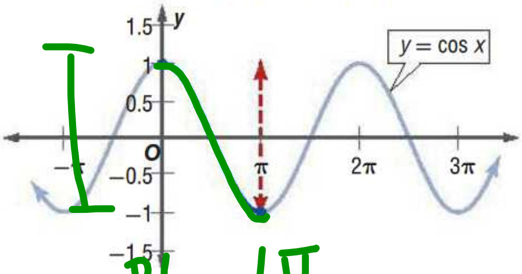
1c. $\sin^{-1}(-2\pi)$ not defined

≈ -1.28

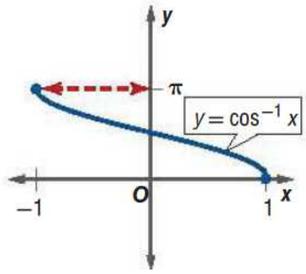
D: $[-1, 1]$

When restricted to a domain of $[0, \pi]$, the cosine function is one-to-one and takes on all of its possible range values on $[-1, 1]$. It is on this restricted domain that the cosine function has an inverse function, called the *inverse cosine function* $y = \cos^{-1} x$ or **arccosine function** $y = \arccos x$. The graph of $y = \cos^{-1} x$ is found by reflecting the graph of the restricted cosine function in the line $y = x$.

Restricted Cosine Function

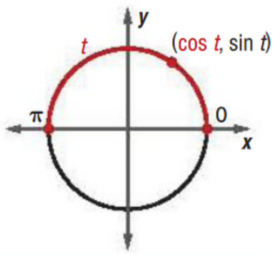


Inverse Cosine Function



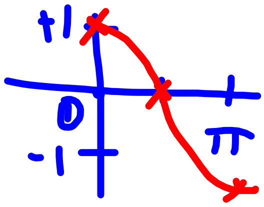
Recall that cos *t* is the *x*-coordinate of the point on the unit circle corresponding to the angle or arc length *t*. Because the range of $y = \cos^{-1} x$ is restricted to $[0, \pi]$, the values of an inverse cosine function are located on the upper half of the unit circle.

Inverse Cosine Values



StudyTip

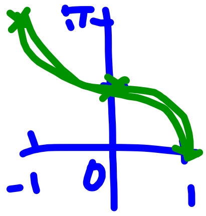
Principal Values Trigonometric functions with restricted domains are sometimes indicated with capital letters. For example, $y = \text{Sin } x$ represents the function $y = \sin x$, where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The values in these restricted domains are often called *principal values*.



$y = \cos x$

D: $[0, \pi]$

R: $[-1, 1]$



$y = \cos^{-1}(x)$

D: $[-1, 1]$

R: $[0, \pi]$

< rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
< deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und

$\arccos \frac{\sqrt{3}}{2}$

$= \frac{\pi}{6}$

A

ref $\frac{\pi}{6}$

$\frac{3\pi}{3} - \frac{\pi}{3}$

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

ref $\frac{\pi}{3}$

EXAMPLE 2 Evaluate Inverse Cosine Functions

A. Find the exact value of $\cos^{-1} 1$, if it exists.

$$2a.\ \cos^{-1}(1) = 0$$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

S | A
T | C
(1, 0)

EXAMPLE 2 Evaluate Inverse Cosine Functions

B. Find the exact value of $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, if it exists.

2b $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\angle$rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\angle$deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

ref$\angle$: $\frac{\pi}{6}$

$\frac{6}{6}\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

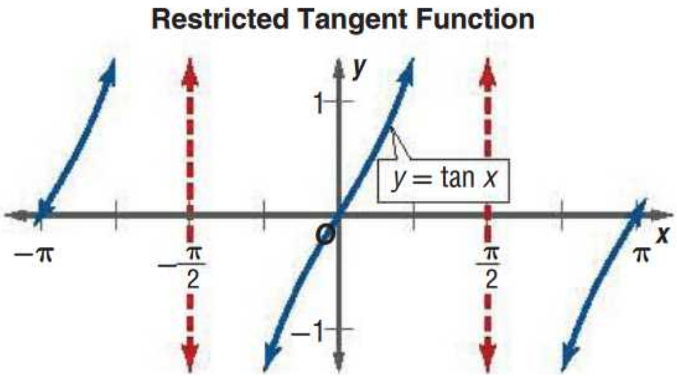
EXAMPLE 2 Evaluate Inverse Cosine Functions

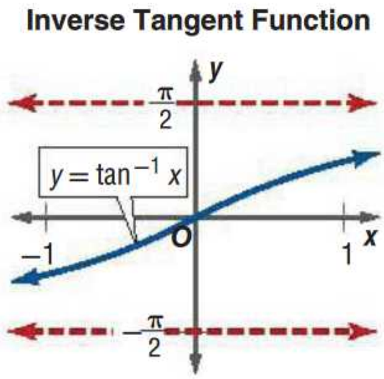
C. Find the exact value of $\cos^{-1}(-2)$, if it exists.

2a. $\cos^{-1}(-2)$ does not exist

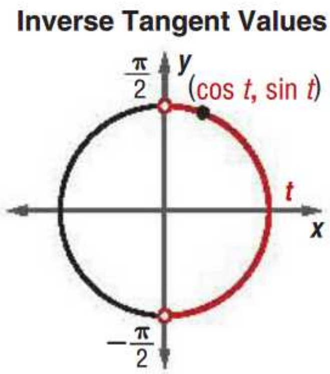
D: $[-1, 1]$

When restricted to a domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the tangent function is one-to-one. It is on this restricted domain that the tangent function has an inverse function called the *inverse tangent function* $y = \tan^{-1} x$ or **arctangent function** $y = \arctan x$. The graph of $y = \tan^{-1} x$ is found by reflecting the graph of the restricted tangent function in the line $y = x$. Notice that unlike the sine and cosine functions, the domain of the inverse tangent function is $(-\infty, \infty)$.





You can also use the unit circle to find the value of an inverse tangent expression. On the unit circle, $\tan t = \frac{\sin t}{\cos t}$ or $\frac{y}{x}$. The values of $y = \tan^{-1} x$ will be located on the right half of the unit circle, not including $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because the tangent function is undefined at those points.

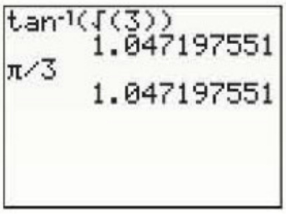


StudyTip

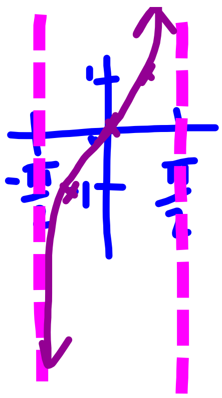
End Behavior of Inverse Tangent Notice that when the graph of the restricted tangent function is reflected in the line $y = x$, the vertical asymptotes at $x = \pm\frac{\pi}{2}$ become the horizontal asymptotes $y = \pm\frac{\pi}{2}$ of the inverse tangent function. Therefore, $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$ and $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$.

TechnologyTip

Evaluate $\tan^{-1}$ You can also use a graphing calculator to find the angle that has a tangent of $\sqrt{3}$.



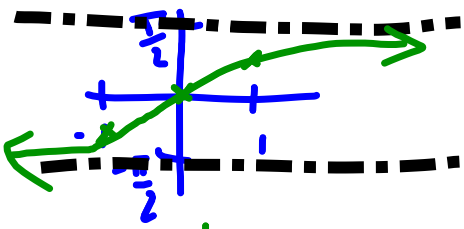
Make sure you select RADIAN on the MODE feature of your graphing calculator.



$y: \tan x$

$D: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$R: (-\infty, \infty)$



$y: \arctan x$

$D: (-\infty, \infty)$

$R: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

$\arctan -\frac{\sqrt{3}}{3}$

$= -\frac{\pi}{6}$

EXAMPLE 3 Evaluate Inverse Tangent Functions

A. Find the exact value of $\tan^{-1}\frac{\sqrt{3}}{3}$, if it exists.

3a $\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 3 Evaluate Inverse Tangent Functions

B. Find the exact value of arctan 1, if it exists.

3b $\arctan(1) = \frac{\pi}{4}$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

While inverse functions for secant, cosecant, and cotangent do exist, these functions are rarely used in computations because the inverse functions for their reciprocals exist. Also, deciding how to restrict the domains of secant, cosecant, and cotangent to obtain arcsecant, arccosecant, and arccotangent is not as apparent. You will explore these functions in Exercise 66.

WatchOut!
Remember that $\pi = 3.14$ radians or 180°.

The three most common inverse trigonometric functions are summarized below.

Key Concept **Inverse Trigonometric Functions**

Inverse Sine of x

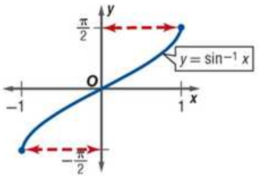
Words The angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of x.

A
C

Symbols $y = \sin^{-1} x$ if and only if $\sin y = x$, for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Inverse Cosine of x

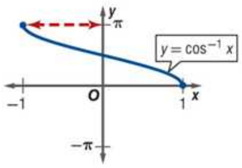
Words The angle (or arc) between 0 and π with a cosine of x.

S | A

Symbols $y = \cos^{-1} x$ if and only if $\cos y = x$, for $-1 \le x \le 1$ and $0 \le y \le \pi$.

Domain: $[-1, 1]$

Range: $[0, \pi]$



Inverse Tangent of x

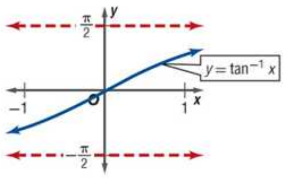
Words The angle (or arc) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a tangent of x.

A
C

Symbols $y = \tan^{-1} x$ if and only if $\tan y = x$, for $-\infty \le x \le \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



You can sketch the graph of one of the inverse trigonometric functions shown above by rewriting the function in the form $\sin y = x$, $\cos y = x$, or $\tan y = x$, assigning values to y and making a table of values, and then plotting the points and connecting the points with a smooth curve.

EXAMPLE 4 Sketch Graphs of Inverse Trigonometric Functions

Sketch the graph of $y = \arctan \frac{x}{2}$.

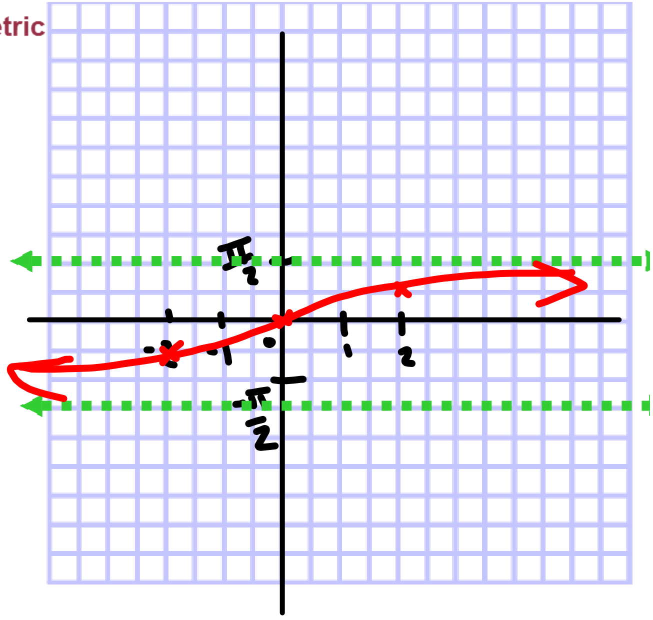
$y = \arctan \frac{x}{2}$

x	y
2	$\frac{\pi}{4}$
0	0
-2	$-\frac{\pi}{4}$

$y = \arctan \frac{x}{2}$

$\tan y = \frac{x}{2}$

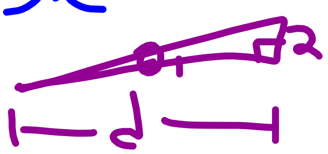
$2\tan y = x$

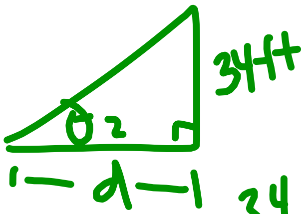


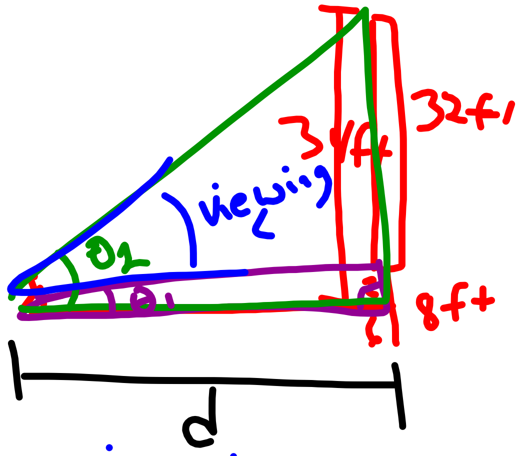
Real-World EXAMPLE 5 Use an Inverse Trigonometric Function

A. MOVIES In a movie theater, a 32-foot-tall screen is located 8 feet above ground. Write a function modeling the viewing angle θ for a person in the theater whose eye-level when sitting is 6 feet above ground.

So







$\tan\theta_1 = \frac{2}{d}$

$\theta_1 = \arctan\left(\frac{2}{d}\right)$

$\tan\theta_2 = \frac{34}{d}$

$\theta_2 = \arctan\left(\frac{34}{d}\right)$

viewing ∠

$\theta = \theta_2 - \theta_1$

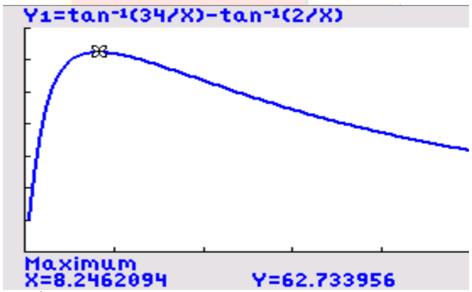
$\theta = \arctan\left(\frac{34}{d}\right) - \arctan\left(\frac{2}{d}\right)$

Real-World EXAMPLE 5 Use an Inverse Trigonometric Function

B. MOVIES In a movie theater, a 32-foot-tall screen is located 8 feet above ground-level. Determine the distance that corresponds to the maximum viewing angle.

$\theta = \arctan\left(\frac{34}{d}\right) - \arctan\left(\frac{2}{d}\right)$

WINDOW
Xmin=0
Xmax=50
Xscl=10
Ymin=0
Ymax=70
Yscl=10



about 8.2 ft