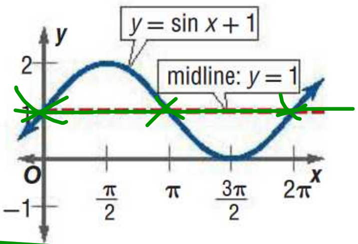
The final way to transform the graph of a sinusoidal function is through a vertical translation or **vertical shift**. Recall from Lesson 1-5 that the graph of $y = f(x) + d$ is the graph of $y = f(x)$ translated or *shifted* $|d|$ units up if $d > 0$ and $|d|$ units down if $d < 0$. The vertical shift is the average of the maximum and minimum values of the function.

The parent functions $y = \sin x$ and $y = \cos x$ oscillate about the x-axis. After a vertical shift, a new horizontal axis known as the **midline** becomes the reference line or equilibrium point about which the graph oscillates. For example, the midline of $y = \sin x + 1$ is $y = 1$, as shown.



In general, the midline for the graphs of $\boxed{y = a \sin (bx + c) + d}$ and $\boxed{y = a \cos (bx + c) + d}$ is $y = d$.

EXAMPLE 6 Graph Vertical Translations of Sinusoidal Functions

State the amplitude, period, frequency, phase shift, and vertical shift of $y = \sin(x + \pi) + 1$. Then graph two periods of the function.

$y = a\sin(bx + c) + d$

$y = \sin(x + \pi) + 1$

amplitude: 1

period $= \frac{2\pi}{|b|} = 2\pi$

frequency $= \frac{1}{2\pi}$

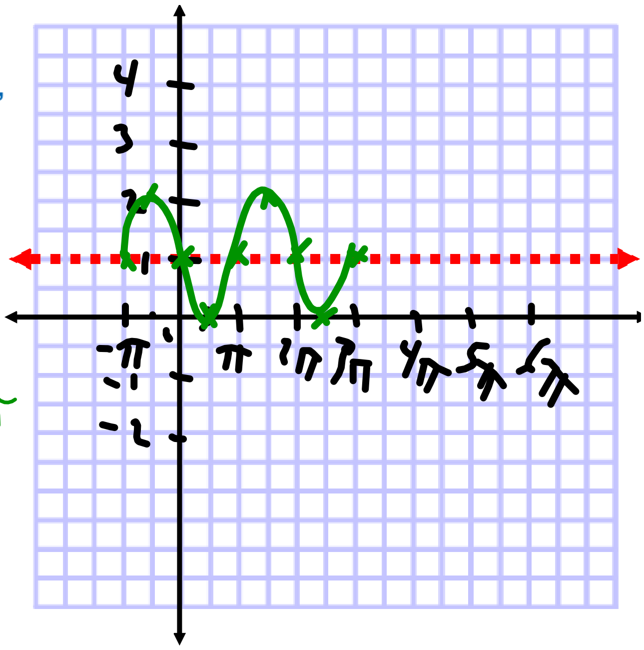
phase shift $-\frac{c}{|b|} = \frac{-\pi}{1} = -\pi$

vertical shift: up 1

start: $x + \pi = 0$, $x = -\pi$

ends: $x + \pi = 2\pi$, $x = \pi$

midline: $y = d$, $y = 1$



The characteristics of transformations of the parent functions $y = \sin x$ and $y = \cos x$ are summarized below.

Concept Summary Graphs of Sinusoidal Functions

The graphs of $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, where $a \neq 0$ and $b \neq 0$, have the following characteristics.

Amplitude: $|a|$

Period: $\frac{2\pi}{|b|}$

Frequency: $\frac{|b|}{2\pi}$ or $\frac{1}{\text{Period}}$

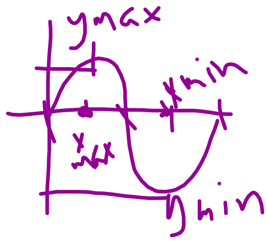
Phase shift: $-\frac{c}{|b|}$

Vertical shift: d

Midline: $y = d$

Technology Tip

Zoom Trig When graphing a trigonometric function using your graphing calculator, be sure you are in radian mode and use the ZTrig selection under the zoom feature to change your viewing window from the standard window to a more appropriate window of $[-2\pi, 2\pi]$ scl: $\pi/2$ by $[-4, 4]$ scl: 1.



$a = \frac{|y_{max} - y_{min}|}{2}$

$d = \frac{(y_{max} + y_{min})}{2}$

$|b| = \frac{2\pi}{\text{period}}$

$\text{period} = 2|x_{max} - x_{min}|$

2 Applications of Sinusoidal Functions Many real-world situations that exhibit periodic behavior over time can be modeled by transformations of $y = \sin x$ or $y = \cos x$.

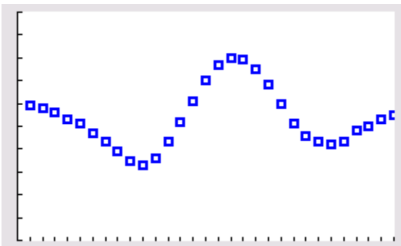
Real-World EXAMPLE 7 Modeling Data Using a Sinusoidal Function

Handout

METEOROLOGY The tides in the Bay of Fundy, in New Brunswick, Canada, have extreme highs and lows everyday. The table shows the high tides for one lunar month. Write a trigonometric function that models the height of the tides as a function of time *x*, where *x* = 1 represents the first day of the month.

Day	High Tide (ft)	Day	High Tide (ft)
1	25.9	16	27.7
2	25.8	17	28.0
3	25.6	18	27.9
4	25.3	19	27.5
5	25.1	20	26.8
6	24.7	21	26.0
7	24.3	22	25.1
8	23.9	23	24.6
9	23.5	24	24.3
10	23.3	25	24.2
11	23.6	26	24.3
12	24.3	27	24.8
13	25.2	28	25.0
14	26.1	29	25.3
15	27.0	30	25.5

1. enter data in L_1 and L_2
2. make scatterplot (decide on scale to see all points)



Real-World EXAMPLE 7 Modeling Data Using a Sinusoidal Function

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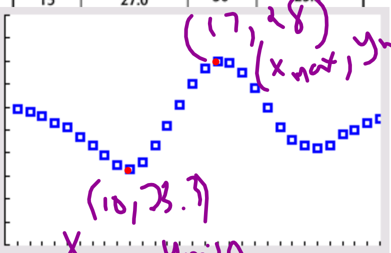
5	25.1	20	26.8
6	24.7	21	26.0
7	24.3	22	25.1
8	23.9	23	24.6
9	23.5	24	24.3
10	23.3	25	24.2
11	23.6	26	24.3
12	24.3	27	24.8
13	25.2	28	25.0
14	26.1	29	25.3
15	27.0	30	25.5



$a = \dfrac{|y_{max} - y_{min}|}{2} = \dfrac{|28 - 23.3|}{2} = 2.35 = a$

use cosine

period $= 2(17-10)$
$= 14$

phase shift
17 right

phase shift $= \dfrac{-c}{|b|}$

$\left(-\dfrac{\pi}{7}\right) 17 = \dfrac{-c}{\frac{\pi}{7}} \left(-\dfrac{\pi}{7}\right)$

$-\dfrac{17\pi}{7} = c$

period $= \dfrac{2\pi}{|b|}$

$14 = \dfrac{2\pi}{|b|}$

$|b| = \dfrac{2\pi}{14}$

$b = \dfrac{\pi}{7}$

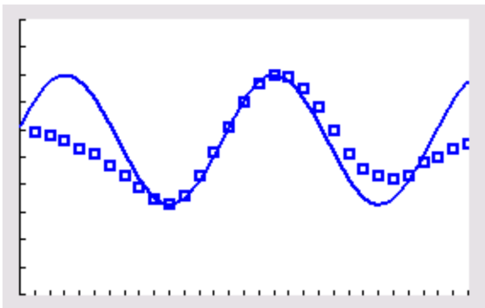
$d = \dfrac{y_{max} + y_{min}}{2}$

$d = \dfrac{28 + 23.3}{2}$

$d = 25.65$

$y = a\cos(bx + c) + d$

$y = 2.35\cos\left(\dfrac{\pi}{7}x - \dfrac{17\pi}{7}\right) + 25.65$



EXAMPLE 1 **Guided Practice**

Describe how the graphs of $f(x) = \cos x$ and $g(x) = 5 \cos x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally.

B. The graph of $g(x)$ is the graph of $f(x)$ compressed vertically.

C. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally.

D. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically.

EXAMPLE 2 Guided Practice

Describe how the graphs of $f(x) = \cos x$ and $g(x) = -6 \cos x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally and then reflected in the y-axis.

B. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and then reflected in the x-axis.

C. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally and then reflected in the x-axis.

D. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and then reflected in the y-axis.

EXAMPLE 3 Guided Practice

Describe how the graphs of $f(x) = \sin x$ and $g(x) = \sin 4x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically.

B. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally.

C. The graph of $g(x)$ is the graph of $f(x)$ compressed vertically.

D. The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally.

Real-World EXAMPLE 4 Guided Practice

MUSIC In the equal tempered scale, F sharp has a frequency of 740 hertz. Write an equation for a sine function that can be used to model the initial behavior of the sound wave associated with F sharp having an amplitude of 0.2.

A. $y = 0.2 \sin 1480\pi t$

B. $y = 0.2 \sin 740\pi t$

C. $y = 0.4 \sin 370\pi t$

D. $y = 0.1 \sin 74\pi t$

EXAMPLE 5 Guided Practice

State the amplitude, period, frequency, and phase shift of $y = 4\cos\left(\frac{x}{3} + \frac{\pi}{6}\right)$.

A. amplitude: 4, period: $\frac{3}{2\pi}$, frequency: $\frac{2\pi}{3}$, phase shift: $-\frac{\pi}{6}$

B. amplitude: $\frac{1}{4}$, period: 3, frequency: $\frac{1}{3}$, phase shift: $-\frac{\pi}{18}$

C. amplitude: 4, period: 6π, frequency: $\frac{1}{6\pi}$, phase shift: $-\frac{\pi}{2}$

D. amplitude: –4, period: $\frac{2\pi}{3}$, frequency: $\frac{3}{2\pi}$, phase shift: $-\frac{\pi}{2}$

EXAMPLE 6 **Guided Practice**

State the amplitude, period, frequency, phase shift, and vertical shift of $y = 3\sin\left(4x + \frac{\pi}{2}\right) - 2$

A. amplitude: 3, period: $\frac{\pi}{4}$, frequency: $\frac{4}{\pi}$, phase shift: $-\frac{\pi}{8}$, vertical shift: 2

B. amplitude: –3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$, phase shift: $-\frac{\pi}{2}$, vertical shift: –2

C. amplitude: 3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$ phase shift: $-\frac{\pi}{4}$, vertical shift: 2

D. amplitude: 3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$, phase shift: $-\frac{\pi}{8}$, vertical shift: –2

Real-World EXAMPLE 7 **Guided Practice**

TEMPERATURES The table shows the average monthly high temperatures for Chicago. Write a function that models the high temperatures using $x = 1$ to represent January.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Temp. (°)	32	38	47	59	70	80	84	83	76	64	49	37

A. $y = 26\cos\left(\frac{\pi}{7}x - \frac{\pi}{7}\right) + 58$

B. $y = 26\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 58$

C. $y = 52\cos\left(\frac{\pi}{6}x - \pi\right) + 26$

D. $y = 23\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 61$

EXAMPLE 1 **Guided Practice**

Describe how the graphs of $f(x) = \cos x$ and $g(x) = 5 \cos x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally.

B. The graph of $g(x)$ is the graph of $f(x)$ compressed vertically.

C. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally.

D. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically.

D.

$f(x) = \cos x$

$g(x) = 5\cos x$

amplitude: 5

EXAMPLE 2 Guided Practice

Describe how the graphs of $f(x) = \cos x$ and $g(x) = -6 \cos x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally and then reflected in the y-axis.

B. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and then reflected in the x-axis.

C. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally and then reflected in the x-axis.

D. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically and then reflected in the y-axis.

$\boxed{B}$

$f(x) = \cos x$

$g(x) = -6 \cos x$

reflect in x-axis

expand vertically

EXAMPLE 3 Guided Practice

Describe how the graphs of $f(x) = \sin x$ and $g(x) = \sin 4x$ are related.

A. The graph of $g(x)$ is the graph of $f(x)$ expanded vertically.

B. The graph of $g(x)$ is the graph of $f(x)$ expanded horizontally.

C. The graph of $g(x)$ is the graph of $f(x)$ compressed vertically.

D. The graph of $g(x)$ is the graph of $f(x)$ compressed horizontally.

$\boxed{D}$

$f(x) = \sin x$ period 2π

$g(x) = \sin 4x$

period: $\frac{2\pi}{4} = \frac{\pi}{2}$

Real-World EXAMPLE 4 **Guided Practice**

MUSIC In the equal tempered scale, F sharp has a frequency of 740 hertz. Write an equation for a sine function that can be used to model the initial behavior of the sound wave associated with F sharp having an amplitude of 0.2.

A. $y = 0.2 \sin 1480\pi t$

B. $y = 0.2 \sin 740\pi t$

C. $y = 0.4 \sin 370\pi t$

D. $y = 0.1 \sin 74\pi t$

$\boxed{A}$

$y = a \sin bt$

$a = .2$

Freq $= 740$

period $= \frac{1}{F}$

$\frac{2\pi}{|b|} = \frac{1}{740}$

$|b| = 2\pi(740)$

$|b| = 1480\pi$

$b = \pm 1480\pi$

let $b = 1480\pi$

$y = .2 \sin 1480\pi t$

EXAMPLE 5 Guided Practice

State the amplitude, period, frequency, and phase shift of $y = 4\cos\left(\frac{x}{3} + \frac{\pi}{6}\right)$.

$\boxed{C}$

A. amplitude: 4, period: $\frac{3}{2\pi}$, frequency: $\frac{2\pi}{3}$, phase shift: $-\frac{\pi}{6}$

B. amplitude: $\frac{1}{4}$, period: 3, frequency: $\frac{1}{3}$, phase shift: $-\frac{\pi}{18}$

C. amplitude: 4, period: 6π, frequency: $\frac{1}{6\pi}$, phase shift: $-\frac{\pi}{2}$

D. amplitude: –4, period: $\frac{2\pi}{3}$, frequency: $\frac{3}{2\pi}$, phase shift: $-\frac{\pi}{2}$

$y = 4\cos\left(\frac{x}{3} + \frac{\pi}{6}\right)$

$y = 4\cos\left(\frac{1}{3}x + \frac{\pi}{6}\right)$

$y = a\cos(bx + c)$

amp $a = 4$

period: $\frac{2\pi}{\frac{1}{3}} = 2\pi \cdot \frac{3}{1} = 6\pi$

freq: $\frac{1}{\text{period}} = \frac{1}{6\pi}$

phase shift: $\frac{-c}{|b|} = \frac{-\frac{\pi}{6}}{\frac{1}{3}} = -\frac{\pi}{6} \cdot \frac{3}{1} = -\frac{\pi}{2}$

EXAMPLE 6 Guided Practice

State the amplitude, period, frequency, phase shift, and vertical shift of $y = 3\sin\left(4x + \frac{\pi}{2}\right) - 2$

$\boxed{D}$

A. amplitude: 3, period: $\frac{\pi}{4}$, frequency: $\frac{4}{\pi}$, phase shift: $-\frac{\pi}{8}$, vertical shift: 2

B. amplitude: –3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$, phase shift: $-\frac{\pi}{2}$, vertical shift: –2

C. amplitude: 3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$ phase shift: $-\frac{\pi}{4}$, vertical shift: 2

D. amplitude: 3, period: $\frac{\pi}{2}$, frequency: $\frac{2}{\pi}$, phase shift: $-\frac{\pi}{8}$, vertical shift: –2

$y = 3\sin\left(4x + \frac{\pi}{2}\right) - 2$

amp = 3

period $\frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$

Freq = $\frac{2}{\pi}$

phaseshift = $-\frac{c}{b} = -\frac{\frac{\pi}{2}}{4} = -\frac{\pi}{2} \cdot \frac{1}{4} = -\frac{\pi}{8}$

vert shift d = -2

Real-World EXAMPLE 7 **Guided Practice**

TEMPERATURES The table shows the average monthly high temperatures for Chicago. Write a function that models the high temperatures using $x = 1$ to represent January.

	1	2	3	4	5	6	7	8	9	10	11	12
Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sept	Oct	Nov	Dec
Temp. (°)	32	38	47	59	70	80	84	83	76	64	49	37

A. $y = 26\cos\left(\frac{\pi}{7}x - \frac{\pi}{7}\right) + 58$

B. $y = 26\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 58$

C. $y = 52\cos\left(\frac{\pi}{6}x - \pi\right) + 26$

D. $y = 23\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 61$

$a = \frac{84-32}{2} = 26$

$d = \frac{84+32}{2} = 58$

period: $2(7-1) = 2 \cdot 6 = 12$

$b = \frac{2\pi}{per} = \frac{2\pi}{12} = \frac{\pi}{6}$

$\boxed{B}$

phase shift max at 7, shift 7 units right

$7 = \frac{-c}{|b|}$

$\left(-\frac{\pi}{6}\right)7 = \frac{-c}{\frac{\pi}{6}}\left(-\frac{\pi}{6}\right)$

$-\frac{7\pi}{6} = c$

$y = a\cos(bx+c)+d$

$y = 26\cos\left(\frac{\pi}{6}x - \frac{7\pi}{6}\right) + 58$

4-4 Graphing Sine & Cosine Functions

EVEN PAGE

TOC

EQ: Can you graph transformations of the sine and cosine functions and use sinusoidal functions to solve problems?

How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...