4-4 Graphing Sine & Cosine Functions

EVEN PAGE

TOC

EQ: Can you graph transformations of the sine and cosine functions and use sinusoidal functions to solve problems?

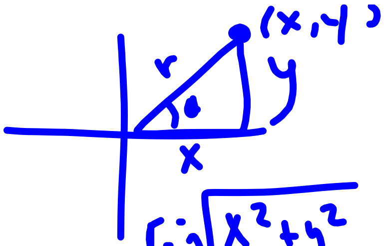
How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...

Day	High Tide (ft)	Day	High Tide (ft)
1	25.9	16	27.7
2	25.8	17	28.0
3	25.6	18	27.9
4	25.3	19	27.5
5	25.1	20	26.8
6	24.7	21	26.0
7	24.3	22	25.1
8	23.9	23	24.6
9	23.5	24	24.3
10	23.3	25	24.2
11	23.6	26	24.3
12	24.3	27	24.8
13	25.2	28	25.0
14	26.1	29	25.3
15	27.0	30	25.5

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

S	A
T	C

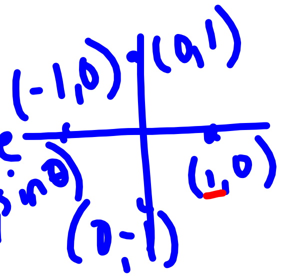


$r = \sqrt{x^2+y^2}$

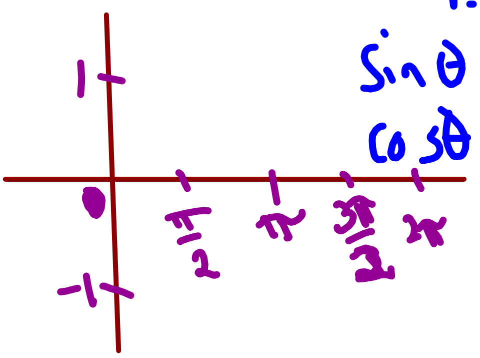
$\sin\theta = \frac{y}{r}$

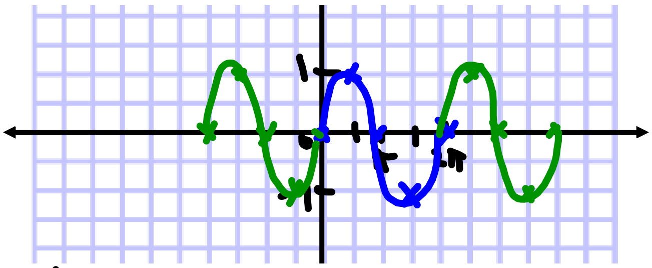
$\cos\theta = \frac{x}{r}$

unit circle

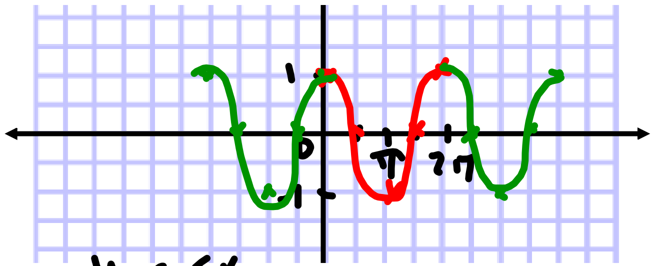


$(\cos\theta, \sin\theta)$





y = sin x



y = cos x

1 Transformations of Sine and Cosine Functions As shown in Explore 4-4, the graph $y = \sin t$ follows the y-coordinate of the point determined by t as it moves around the unit ircle. Similarly, the graph of $y = \cos t$ follows the x-coordinate of this point. The graphs of these unctions are periodic, repeating after a period of 2π. The properties of the sine and cosine unctions are summarized below.

Key Concept **Properties of the Sine and Cosine Functions**

Sine Function

Domain: $(-\infty, \infty)$ **Range:** $[-1, 1]$

y-intercept: 0

x-intercepts: $n\pi$, $n \in \mathbb{Z}$ $0 + \pi n$

Continuity: continuous on $(-\infty, \infty)$

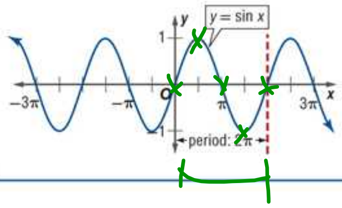
Symmetry: origin (odd function)

Extrema: maximum of 1 at $x = \frac{\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$

minimum of -1 at $x = \frac{3\pi}{2} + 2n\pi$, $n \in \mathbb{Z}$

End Behavior: $\lim_{x \to -\infty} \sin x$ and $\lim_{x \to \infty} \sin x$ do not exist.

Oscillation: between -1 and 1



Cosine Function

Domain: $(-\infty, \infty)$ **Range:** $[-1, 1]$

y-intercept: 1

x-intercepts: $\frac{\pi}{2}n$, $n \in \mathbb{Z}$ $\frac{\pi}{2} + \pi n$ $n \in \mathbb{Z}$

Continuity: continuous on $(-\infty, \infty)$

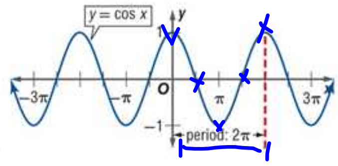
Symmetry: y-axis (even function)

Extrema: maximum of 1 at $x = 2n\pi$, $n \in \mathbb{Z}$ $2\pi n$

minimum of -1 at $x = \pi + 2n\pi$, $n \in \mathbb{Z}$ $x = \pi + 2\pi n$

End Behavior: $\lim_{x \to -\infty} \cos x$ and $\lim_{x \to \infty} \cos x$ do not exist.

Oscillation: between -1 and 1

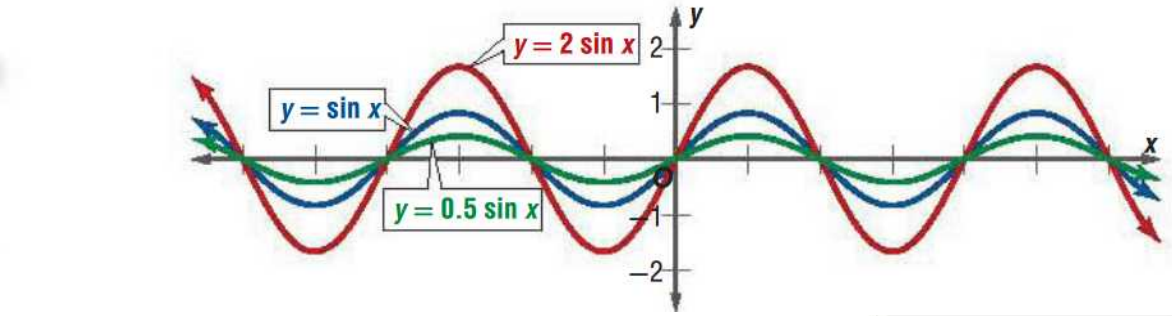


The portion of each graph on $[0, 2\pi]$ represents one period or *cycle* of the function. Notice that the cosine graph is a horizontal translation of the sine graph. Any transformation of a sine function is called a **sinusoid**. The general form of the sinusoidal functions sine and cosine are

$$y = a \sin (bx + c) + d \quad \text{and} \quad y = a \cos (bx + c) + d$$

where a, b, c, and d are constants and neither a nor b is 0.

Notice that the constant factor a in $y = a \sin x$ and $y = a \cos x$ expands the graphs of $y = \sin x$ and $y = \cos x$ vertically if $|a| > 1$ and compresses them vertically if $|a| < 1$.



Vertical dilations affect the *amplitude* of sinusoidal functions.

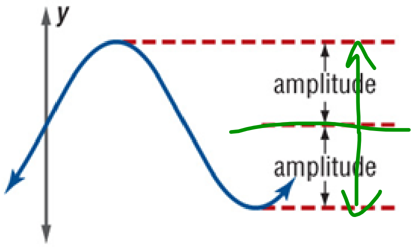
StudyTip

Dilations and x-intercepts Notice that a dilation of a sinusoidal function does not affect where the curve crosses the x-axis, at its x-intercepts.

Key Concept Amplitudes of Sine and Cosine Functions

Words The **amplitude** of a sinusoidal function is half the distance between the maximum and minimum values of the function or half the height of the wave.

Model



Symbols For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, amplitude $= |a|$.

To graph a sinusoidal function of the form $y = a \sin x$ or $y = a \cos x$, plot the x-intercepts of the parent sine or cosine function and use the amplitude $|a|$ to plot the new maximum and minimum points. Then sketch the sine wave through these points.

StudyTip

Radians Versus Degrees You could rescale the x-axis in terms of degrees and produce sinusoidal graphs that look similar to those produced using radian measure. In calculus, however, you will encounter rules that depend on radian measure. So, in this book, we will graph all trigonometric functions in terms of radians.

EXAMPLE 1 Graph Vertical Dilations of Sinusoidal Functions

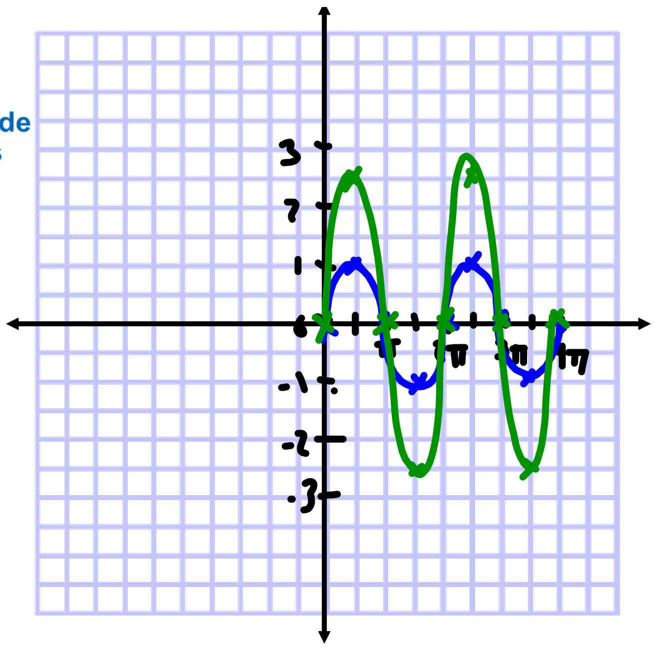
Describe how the graphs of $f(x) = \sin x$ and $g(x) = 2.5 \sin x$ are related. ~~Then find the~~ amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.

$f(x) = \sin x$

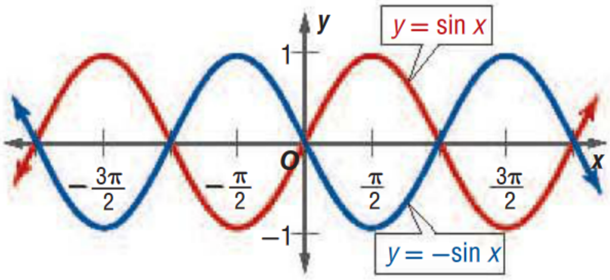
$g(x) = 2.5 \sin x$

amplitude $a = 2.5$

Vertical stretch by factor 2.5



If $a < 0$, the graph of the sinusoidal function is reflected in the x-axis.



EXAMPLE 2 Graph Reflections of Sinusoidal Functions

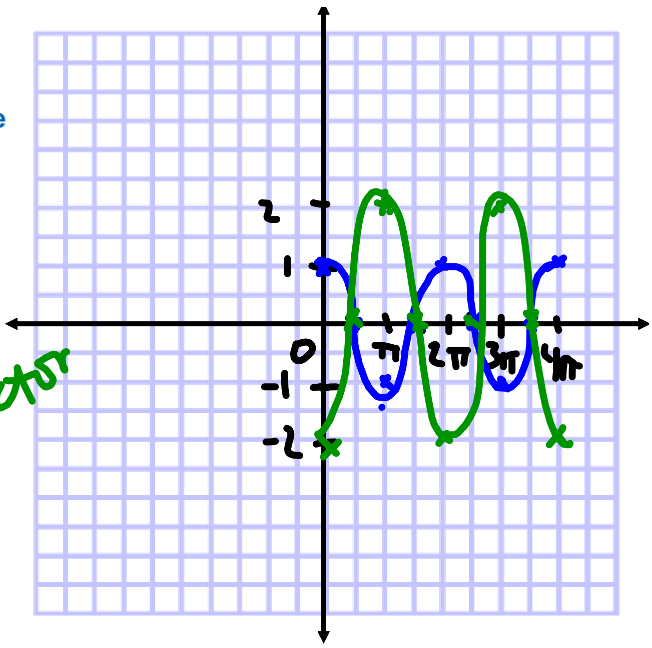
Describe how the graphs of $f(x) = \cos x$ and $g(x) = -2 \cos x$ are related. Then find the amplitude of $g(x)$, and sketch two periods of both functions on the same coordinate axes.

$f(x) = \cos x$

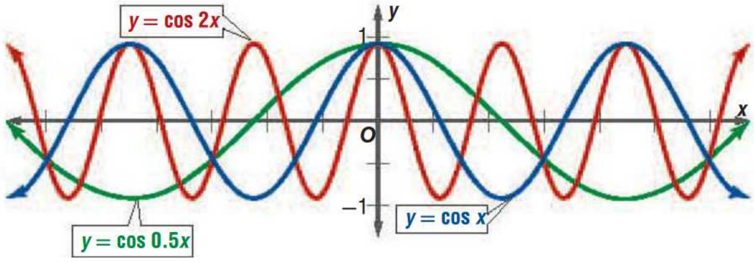
$g(x) = -2\cos x$

reflect in x-axis & stretch vertically by factor of 2

$a = 2$



In Lesson 1-5, you learned that if $g(x) = f(bx)$, then $g(x)$ is the graph of $f(x)$ compressed horizontally if $|b| > 1$ and expanded horizontally if $|b| < 1$. Horizontal dilations affect the *period* of a sinusoidal function—the length of one full cycle.

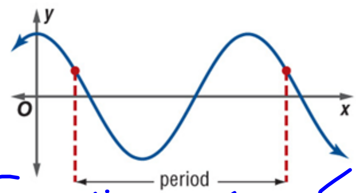


Key Concept **Periods of Sine and Cosine Functions**

Words The period of a sinusoidal function is the distance between any two sets of repeating points on the graph of the function.

Symbols For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, where $b \neq 0$, period $= \frac{2\pi}{|b|}$.

Model

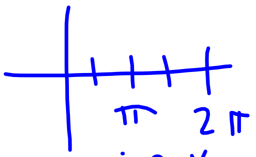


perid $= \frac{2\pi}{|b|}$ for sin & cos

To graph a sinusoidal function of the form $y = \sin bx$ or $y = \cos bx$, find the period of the function and successively add $\frac{\text{period}}{4}$ to the left endpoint of an interval with that length. Then use these values as the x-values for the key points on the graph.

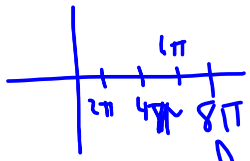
WatchOut!

Determining Period When determining the period of a periodic function from its graph, remember that the period is the *smallest* distance that contains all values of the function.



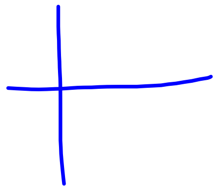
y = sin x

period: 2π



period

8π



EXAMPLE 3 Graph Horizontal Dilations of Sinusoidal Functions

Describe how the graphs of $f(x) = \cos x$ and $g(x) = \cos \frac{x}{8}$ are related. Then find the period of $g(x)$, and sketch at least one period of both functions on the same coordinate axes.

$f(x) = \cos x$

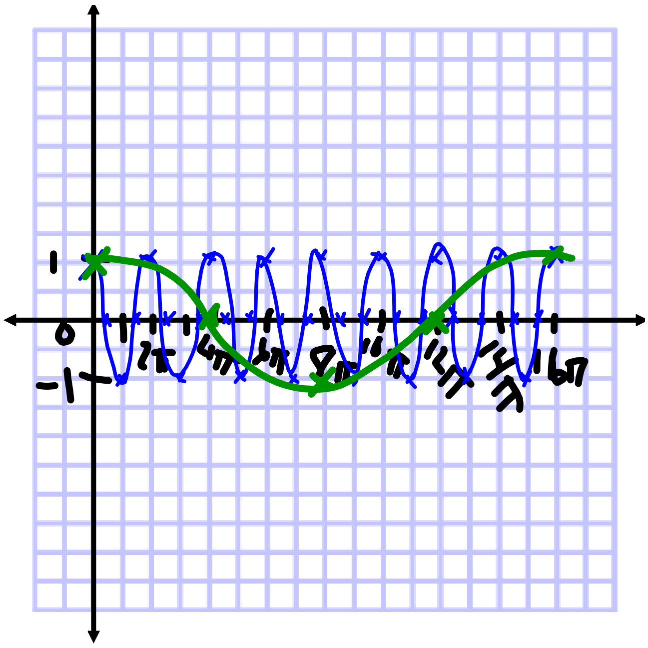
$g(x) = \cos \frac{x}{8} = \cos \frac{1}{8}x$

$b = \frac{1}{8}$

period: $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{8}} = 2\pi \cdot \frac{8}{1} = 16\pi$

4 cubby holes

$\frac{16\pi}{4} = 4\pi$



Horizontal dilations also affect the *frequency* of sinusoidal functions.

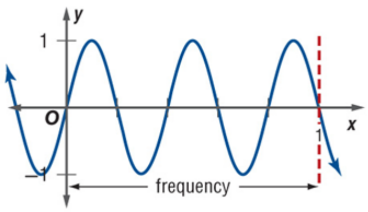
Key Concept **Frequency of Sine and Cosine Functions**

Words The **frequency** of a sinusoidal function is the number of cycles the function completes in a one unit interval. The frequency is the reciprocal of the period.

Symbols For $y = a \sin(bx + c) + d$ and $y = a \cos(bx + c) + d$,

$$\text{frequency} = \frac{1}{\text{period}} \text{ or } \frac{|b|}{2\pi}.$$

Model



Because the frequency of a sinusoidal function is the reciprocal of the period, it follows that the period of the function is the reciprocal of its frequency.

Real-World EXAMPLE 4 Use Frequency to Write a Sinusoidal Function

MUSIC A bass tuba can hit a note with a frequency of 50 cycles per second (50 hertz) and an amplitude of 0.75. Write an equation for a cosine function that can be used to model the initial behavior of the sound wave associated with the note.

$y = a\cos bt$

$a = .75$

$F = \frac{1}{\text{period}}$

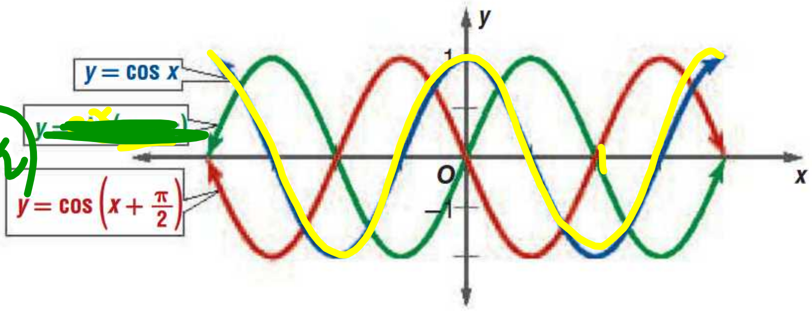
$F = \frac{b}{2\pi}$

$2\pi \cdot 50 = \frac{b}{2\pi} \cdot 2\pi$

$100\pi = b$

$y = .75\cos 100\pi t$

A *phase* of a sinusoid is the position of a wave relative to some reference point. A horizontal translation of a sinusoidal function results in a *phase shift*. Recall from Lesson 1-5 that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ translated or *shifted* $|c|$ units left if $c > 0$ and $|c|$ units right if $c < 0$.

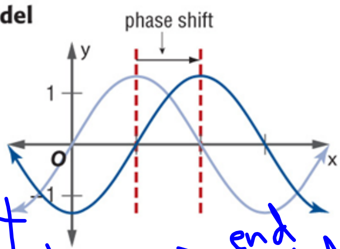


green
$y = \cos(x - \frac{\pi}{2})$

Key Concept — Phase Shift of Sine and Cosine Functions

Words The **phase shift** of a sinusoidal function is the difference between the horizontal position of the function and that of an otherwise similar sinusoidal function.

Model



Symbols For $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$, where $b \neq 0$,

phase shift $= -\frac{c}{|b|}$.

$-\frac{c}{|b|}$

start period $bx+c=0$

end period $bx+c=2\pi$

To graph the phase shift of a sinusoidal function of the form $y = a \sin (bx + c) + d$ or $y = a \cos (bx + c) + d$, first determine the endpoints of an interval that corresponds to one cycle of the graph by adding $-\frac{c}{b}$ to each endpoint on the interval $[0, 2\pi]$ of the parent function.

StudyTip

Alternative Form The general forms of the sinusoidal functions can also be expressed as $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$. In these forms, each sinusoid has a phase shift of h and a vertical translation of k in comparison to the graphs of $y = a \sin bx$ and $y = a \cos bx$.

EXAMPLE 5 Graph Horizontal Translations of Sinusoidal Functions

$y = a\sin(bx+c)+d$

State the amplitude, period, frequency, and phase shift of $y = 2\sin\left(5x + \frac{\pi}{4}\right)$. Then graph two periods of the function.

$y = 2\sin\left(5x + \frac{\pi}{4}\right)$

amplitude: 2

period: $\frac{2\pi}{5}$

frequency: $\frac{5}{2\pi}$

phase shift: $\frac{-c}{|b|} = \frac{-\frac{\pi}{4}}{5} = \frac{-\pi}{4} \cdot \frac{1}{5} = \frac{-\pi}{20}$

Start

$5x + \frac{\pi}{4} = 0$

$5x = -\frac{\pi}{4}$

$x = -\frac{\pi}{20}$

end

$5x + \frac{\pi}{4} = \frac{8\pi}{4}$

$5x = \frac{7\pi}{4}$

$x = \frac{7\pi}{20}$