4-3 Trigonometric Functions on the Unit Circle

EVEN PAGE

TOC

EQ: Can you find values of trigonometric functions for any angle, and can you find them using the unit circle?

How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...

New Vocabulary

- quadrantal angle
- reference angle
- unit circle
- circular function
- periodic function
- period

1 Trigonometric Functions of Any Angle

In Lesson 4-1, the definitions of the six trigonometric functions were restricted to positive acute angles. In this lesson, these definitions are extended to include *any* angle.

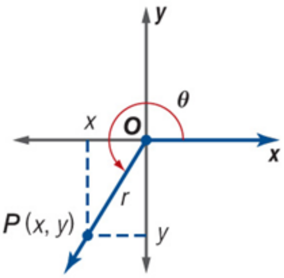
Key Concept — **Trigonometric Functions of Any Angle**

Let θ be any angle in standard position and point $P(x, y)$ be a point on the terminal side of θ. Let r represent the nonzero distance from P to the origin. That is, let $r = \sqrt{x^2 + y^2} \neq 0$. Then the trigonometric functions of θ are as follows.

$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}, y \neq 0$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}, x \neq 0$

$\tan\theta = \frac{y}{x}, x \neq 0$ $\cot\theta = \frac{x}{y}, y \neq 0$



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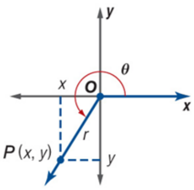
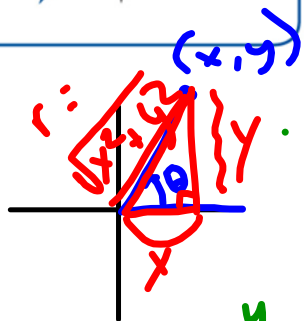


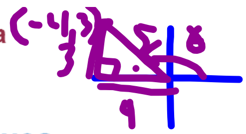


$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

EXAMPLE 1 Evaluate Trigonometric Functions Given a Point



Let (–4, 3) be a point on the terminal side of an angle θ in standard position. Find the exact values of the six trigonometric functions of θ.

① $(-4, 3)$ → $x = -4$, $y = 3$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$r = 5$

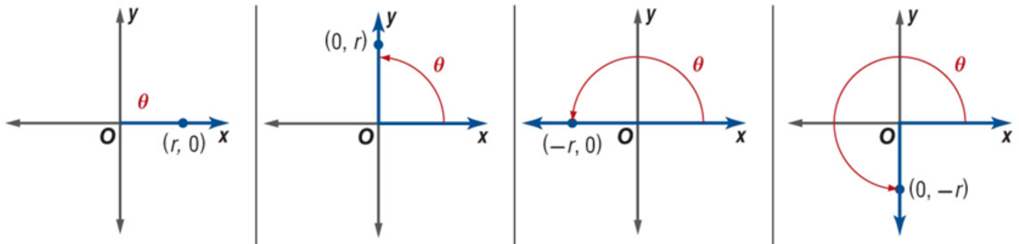
$\sin\theta = \frac{y}{r} = \frac{3}{5}$ $\qquad$ $\csc\theta = \frac{5}{3}$

$\cos\theta = \frac{x}{r} = -\frac{4}{5}$ $\qquad$ $\sec\theta = -\frac{5}{4}$

$\tan\theta = \frac{y}{x} = -\frac{3}{4}$ $\qquad$ $\cot\theta = -\frac{4}{3}$

In Example 1, you found the trigonometric values of θ without knowing the measure of θ. Now we will discuss methods for finding these function values when only θ is known. Consider trigonometric functions of quadrantal angles. When the terminal side of an angle θ that is in standard position lies on one of the coordinate axes, the angle is called a **quadrantal angle**.

Key Concept Common Quadrantal Angles



$\theta = 0°$ or 0 radians | $\theta = 90°$ or $\frac{\pi}{2}$ radians | $\theta = 180°$ or π radians | $\theta = 270°$ or $\frac{3\pi}{2}$ radians

0° = 0
(1,0) r = 1

(0,1)
r = 1
90° = $\frac{\pi}{2}$

(-1,0) 180° = π

270° = $\frac{3\pi}{2}$
(0,-1)

You can find the values of the trigonometric functions of quadrantal angles by choosing a point on the terminal side of the angle and evaluating the function at that point. Any point can be chosen. However, to simplify calculations, pick a point for which r equals 1.

StudyTip

Quadrantal Angles There are infinitely many quadrantal angles that are coterminal with the quadrantal angles listed at the right. The measure of a quadrantal angle is a multiple of 90° or $\frac{\pi}{2}$.

EXAMPLE 2 Evaluate Trigonometric Functions of Quadrantal Angles

A. Find the exact value of cos π. If not defined, write *undefined*.

2a $\cos \pi$

$(-1, 0)$
$x \quad y$
$r = 1$

$\cos\theta = \frac{x}{r}$

$\cos\pi = \frac{-1}{1} = -1$

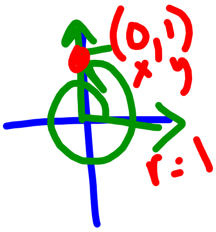
EXAMPLE 2 Evaluate Trigonometric Functions of Quadrantal Angles

B. Find the exact value of tan 450°. If not defined, write *undefined*.

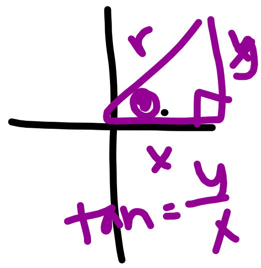
2b $\tan 450°$

$\tan\theta = \frac{y}{x}$

$\tan 450° = \frac{1}{0} =$ undefined



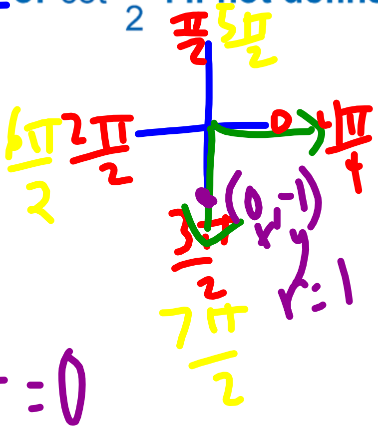
reminder



EXAMPLE 2 Evaluate Trigonometric Functions of Quadrantal Angles

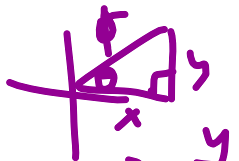
C. Find the exact value of $\cot \frac{7\pi}{2}$. If not defined, write *undefined.*

2c) $\cot \frac{7\pi}{2}$



$\cot\theta = \frac{x}{y}$

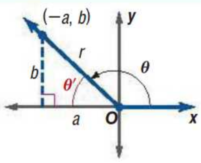
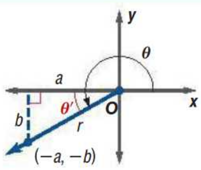
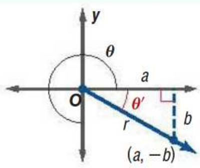
$\cot \frac{7\pi}{2} = \frac{0}{-1} = 0$



$\tan\theta = \frac{y}{x}$

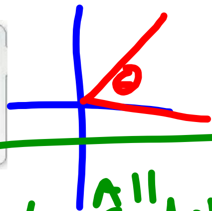
$\cot\theta = \frac{x}{y}$

To find the values of the trigonometric functions of angles that are neither acute nor quadrantal, consider the three cases shown below in which a and b are positive real numbers. Compare the values of sine, cosine, and tangent of θ and θ'.

Quadrant II		Quadrant III		Quadrant IV	
					
$\sin\theta = \frac{b}{r}$	$\sin\theta' = \frac{b}{r}$	$\sin\theta = -\frac{b}{r}$	$\sin\theta' = \frac{b}{r}$	$\sin\theta = -\frac{b}{r}$	$\sin\theta' = \frac{b}{r}$
$\cos\theta = -\frac{a}{r}$	$\cos\theta' = \frac{a}{r}$	$\cos\theta = -\frac{a}{r}$	$\cos\theta' = \frac{a}{r}$	$\cos\theta = \frac{a}{r}$	$\cos\theta' = \frac{a}{r}$
$\tan\theta = -\frac{b}{a}$	$\tan\theta' = \frac{b}{a}$	$\tan\theta = \frac{b}{a}$	$\tan\theta' = \frac{b}{a}$	$\tan\theta = -\frac{b}{a}$	$\tan\theta' = \frac{b}{a}$

StudyTip

Reference Angles Notice that in some cases, the three trigonometric values of θ and θ' (read *theta prime*) are the same. In other cases, they differ only in sign.



S	A
T	C

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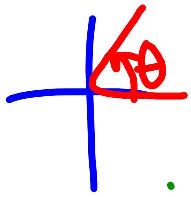
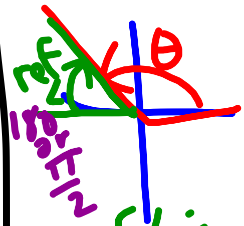
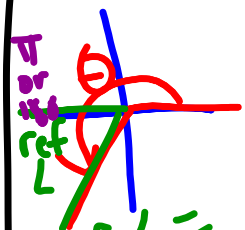
This angle θ', called a **reference angle**, can be used to find the trigonometric values of any angle θ. ~~To find a reference angle for angles~~ outside the interval $0° < \theta < 360°$ or $0 < \theta < 2\pi$, first find a corresponding coterminal angle in this interval.



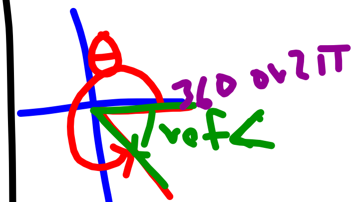
ref∠ = orig ∠



ref∠ = 180 − θ or π − θ



ref∠ = θ − 180 or θ − π

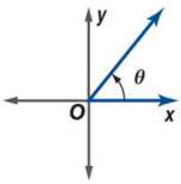


ref∠ = 360 − θ or 2π − θ

Key Concept **Reference Angle Rules**

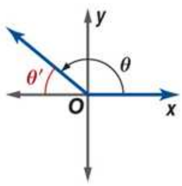
If θ is an angle in standard position, its reference angle θ' is the acute angle formed by the terminal side of θ and the x-axis. The reference angle θ' for any angle θ, $0° < \theta < 360°$ or $0 < \theta < 2\pi$, is defined as follows.

Quadrant I



$\theta' = \theta$

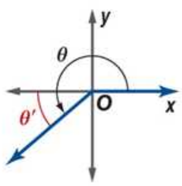
Quadrant II



$\theta' = 180° - \theta$

$\theta' = \pi - \theta$

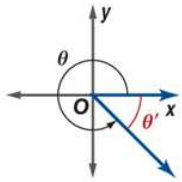
Quadrant III



$\theta' = \theta - 180°$

$\theta' = \theta - \pi$

Quadrant IV



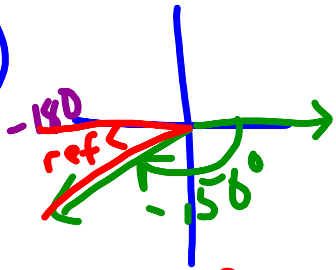
$\theta' = 360° - \theta$

$\theta' = 2\pi - \theta$

EXAMPLE 3 Find Reference Angles

A. Sketch –150°. Then find its reference angle.

3a

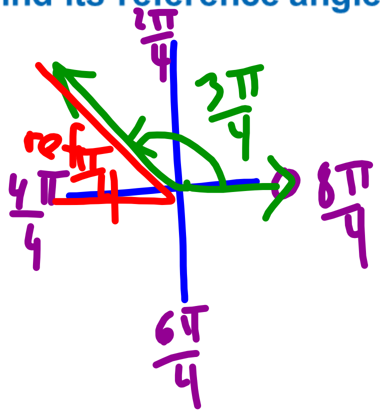


ref < = 30°

EXAMPLE 3 Find Reference Angles

B. Sketch $\frac{3\pi}{4}$. Then find its reference angle.

36 $\frac{3\pi}{4}$



$\text{ref}\angle = \pi - \frac{3\pi}{4}$

$= \frac{4\pi}{4} - \frac{3\pi}{4}$

$= \frac{\pi}{4}$

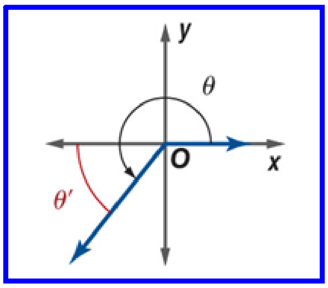
Because the trigonometric values of an angle and its reference angle are equal or differ only in sign, you can use the following steps to find the value of a trigonometric function of any angle θ.

Key Concept **Evaluating Trigonometric Functions of Any Angle**

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.



The signs of the trigonometric functions in each quadrant can be determined using the function definitions given on page 242. For example, because $\sin\theta = \frac{y}{r}$, it follows that $\sin\theta$ is negative when $y < 0$, which occurs in Quadrants III and IV. Using this same logic, you can verify each of the signs for $\sin\theta$, $\cos\theta$, and $\tan\theta$ shown in the diagram. Notice that these values depend only on x and y because r is always positive.

Quadrant II $\sin\theta$: $+$ $\cos\theta$: $-$ $\tan\theta$: $-$	**Quadrant I** $\sin\theta$: $+$ $\cos\theta$: $+$ $\tan\theta$: $+$
Quadrant III $\sin\theta$: $-$ $\cos\theta$: $-$ $\tan\theta$: $+$	**Quadrant IV** $\sin\theta$: $-$ $\cos\theta$: $+$ $\tan\theta$: $-$

x

Ask me about: All Students Take Calculus

Because you know the exact trigonometric values of 30°, 45°, and 60° angles, you can find the exact trigonometric values of *all* angles for which these angles are reference angles. The table lists these values for θ in both degrees and radians.

StudyTip

Memorizing Trigonometric Values

To memorize the exact values of sine for 0°, 30°, 45°, 60°, and 90°, consider the following pattern.

$\sin 0° = \frac{\sqrt{0}}{2}$, or 0

$\sin 30° = \frac{\sqrt{1}}{2}$, or $\frac{1}{2}$

$\sin 45° = \frac{\sqrt{2}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$

$\sin 90° = \frac{\sqrt{4}}{2}$, or 1

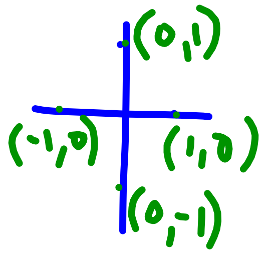
A similar pattern exists for the cosine function, except the values are given in reverse order.

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Ask me about generating a better table

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

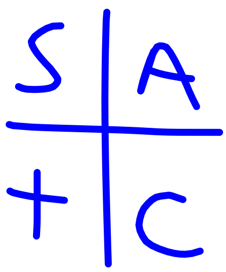
$\tan\theta = \frac{\sin\theta}{\cos\theta}$



$r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$



∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 4 Use Reference Angles to Find Trigonometric Values

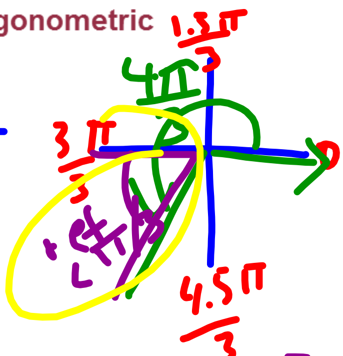
A. Find the exact value of $\sin\frac{4\pi}{3}$.

4a) Exact value $\sin\frac{4\pi}{3}$

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$-\sin\frac{\pi}{3}$

$-\frac{\sqrt{3}}{2}$



ref ∠: $\frac{4\pi}{3} - \frac{3\pi}{3} = \frac{\pi}{3}$

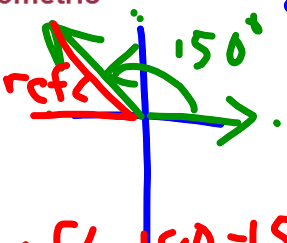
∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

EXAMPLE 4 Use Reference Angles to Find Trigonometric Values

B. Find the exact value of tan 150°.

4B $\tan 150^\circ = -\tan 30^\circ$

$= -\dfrac{\sqrt{3}}{3}$



sin is +

tan θ is neg

ref∠: 180 − 150

ref∠ = 30°

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und

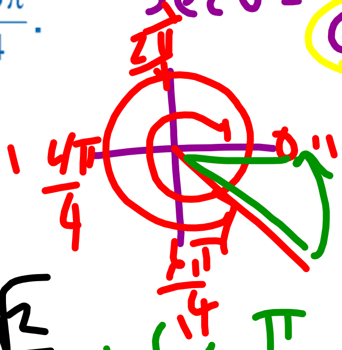
−

EXAMPLE 4 Use Reference Angles to Find Trigonometric Values

C. Find the exact value of $\sec \frac{15\pi}{4}$.

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\sec \frac{15\pi}{4} = \frac{1}{\cos \frac{15\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = 1 \cdot \frac{2}{\sqrt{2}} = \frac{2 \cdot \sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$



ref ∠: $\frac{\pi}{4}$

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

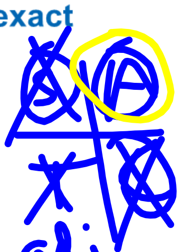
If the value of one or more of the trigonometric functions and the quadrant in which the terminal side of θ lies is known, the remaining function values can be found.

EXAMPLE 5 Use One Trigonometric Value to Find Others

Let $\sec\theta = \frac{\sqrt{29}}{5}$, where $\sin\theta > 0$. Find the exact values of the remaining five trigonometric functions of θ.

WatchOut!

Rationalizing the Denominator Be sure to rationalize the denominator, if necessary.

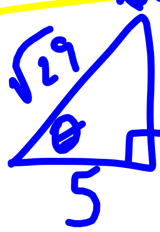


5 $\sec\theta = \frac{\sqrt{29}}{5}$ $\quad \sin\theta > 0$

$\sec\theta = \frac{1}{\cos\theta}$ $\quad \cos\theta > 0$

$\cos\theta = \frac{adj}{hyp}$

$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{29}}{5}$



$\sqrt{(\sqrt{29})^2 - 5^2}$

$\sqrt{29-25}$

$\sqrt{4} = 2$

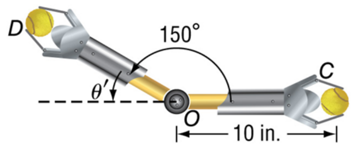
$\sin\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$ $\qquad \csc\theta = \frac{\sqrt{29}}{2}$

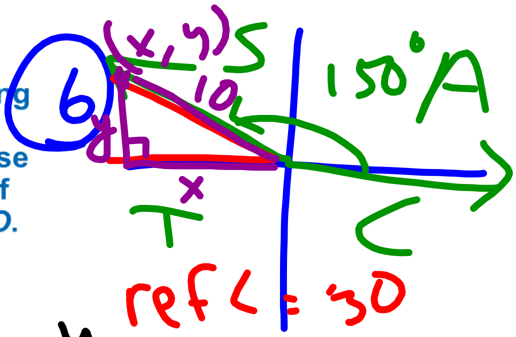
$\cos\theta = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$ $\qquad \sec\theta = \frac{\sqrt{29}}{5}$

$\tan\theta = \frac{2}{5}$ $\qquad \cot\theta = \frac{5}{2}$

Real-World EXAMPLE 6 Find Coordinates Given a Radius and an Angle

ROBOTICS A student programmed a 10-inch long robotic arm to pick up an object at point *C* and rotate through an angle of 150° in order to release it into a container at point *D*. Find the position of the object at point *D*, relative to the pivot point *O*.





$\sin\theta = \frac{y}{r}$

$\sin 150 = \frac{y}{10}$

$\frac{1}{2} = \frac{y}{10}$

$2y = 10$

$y = 5$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
sinθ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tanθ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

$\cos\theta = \frac{x}{r}$

$\cos 150 = \frac{x}{10}$

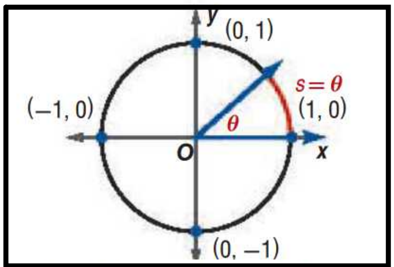
$-\frac{\sqrt{3}}{2} = \frac{x}{10}$

$-\frac{10\sqrt{3}}{2} = x$

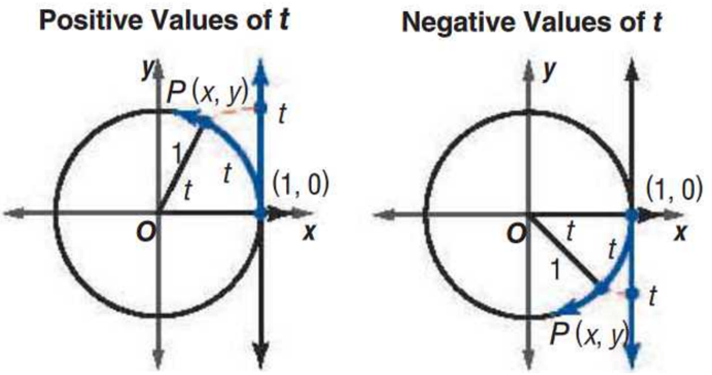
$-5\sqrt{3} = x$

$(-5\sqrt{3}, 5)$

2 Trigonometric Functions on the Unit Circle A unit circle is a circle of radius 1 centered at the origin. Notice thaton a unit circle, the radian measure of a central angle $\theta = \frac{s}{1}$ or s, so the arc length intercepted by θ corresponds exactly to the angle's radian measure. This provides a way of mapping a real number input value for a trigonometric function to a real number output value.



Consider the real number line placed vertically tangent to the unit circle at (1, 0) as shown below. If this line were wrapped about the circle in both the positive (counterclockwise) and negative (clockwise) direction, each point t on the line would map to a unique point $P(x, y)$ on the circle. Because $r = 1$, we can define the trigonometric ratios of angle t in terms of just x and y.



StudyTip

Wrapping Function The association of a point on the number line with a point on a circle is called the *wrapping function*, $w(t)$. For example, if $w(t)$ associates a point t on the number line with a point $P(x, y)$ on the unit circle, then $w(\pi) = (-1, 0)$ and $w(2\pi) = (1, 0)$.

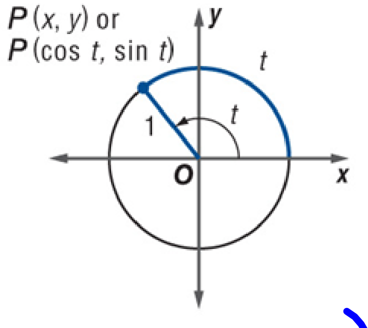
Key Concept Trigonometric Functions on the Unit Circle

Let t be any real number on a number line and let $P(x, y)$ be the point on t when the number line is wrapped onto the unit circle. Then the trigonometric functions of t are as follows.

$\sin t = y$ $\qquad \cos t = x$ $\qquad \tan t = \frac{y}{x}, x \neq 0$

$\csc t = \frac{1}{y}, y \neq 0$ $\qquad \sec t = \frac{1}{x}, x \neq 0$ $\qquad \cot t = \frac{x}{y}, y \neq 0$

Therefore, the coordinates of P corresponding to the angle t can be written as $P(\cos t, \sin t)$.



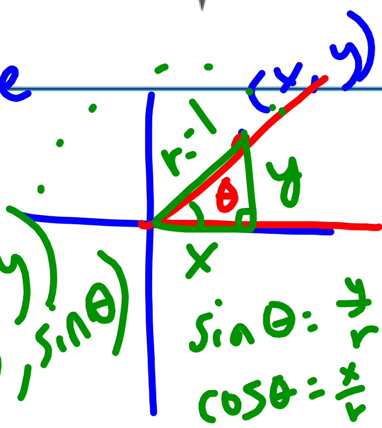
$\sin t = y \qquad \csc t = \frac{1}{y}$

$\cos t = x \qquad \sec t = \frac{1}{x}$

$\tan t = \frac{y}{x} \qquad \cot t = \frac{x}{y}$

$(x, y) = (\cos t, \sin t)$

unit circle
$r = 1$



(x, y)
$(\cos\theta, \sin\theta)$

$\sin\theta = \frac{y}{r} = y$

$\cos\theta = \frac{x}{r} = x$

Notice that the input value in each of the definitions above can be thought of as an angle measure or as a real number t. When defined as functions of the real number system using the unit circle, the trigonometric functions are often called **circular functions**.

Using reference angles or quadrantal angles, you should now be able to find the trigonometric function values for all integer multiples of 30°, or $\frac{\pi}{6}$ radians, and 45°, or $\frac{\pi}{4}$ radians. These special values wrap to 16 special points on the unit circle, as shown below.

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

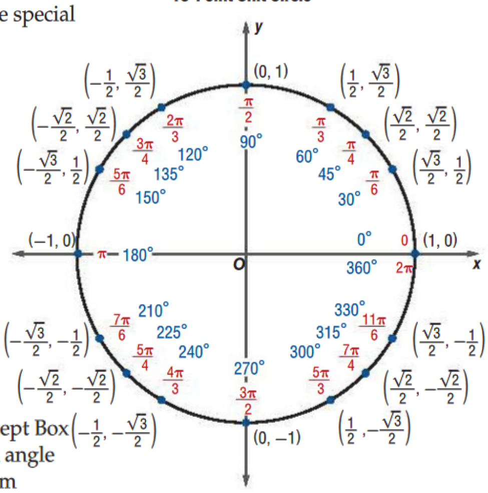
S | A
T | C

unit circle
$(\cos\theta, \sin\theta)$

StudyTip

16-Point Unit Circle You have already memorized these values in the first quadrant. The remaining values can be determined using the x-axis, y-axis, and origin symmetry of the unit circle along with the signs of x and y in each quadrant.

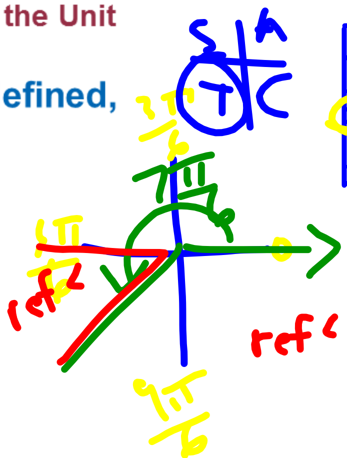
16-Point Unit Circle



Using the (x, y) coordinates in the 16-point unit circle and the definitions in the Key Concept Box at the top of the page, you can find the values of the trigonometric functions for common angle measures. It is helpful to memorize these exact function values so you can quickly perform calculations involving them.

EXAMPLE 7 Find Trigonometric Values Using the Unit Circle

A. Find the exact value of $\sin\frac{7\pi}{6}$. If undefined, write *undefined*.



∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und

7A Exact value $\sin\frac{7\pi}{6}$

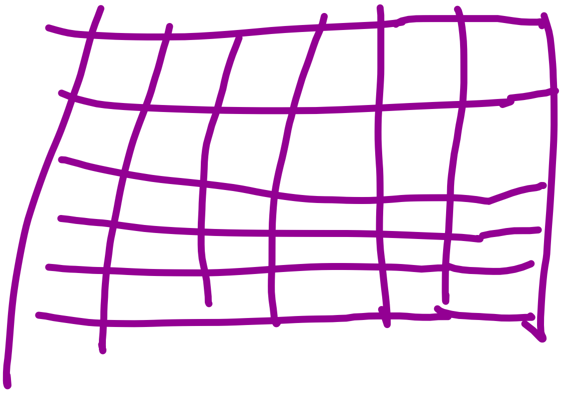
$$\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6}$$

$$= -\frac{1}{2}$$

ref ∠ $= \frac{7\pi}{6} - \frac{\pi}{6}$

$= \frac{\pi}{6}$

neg



$\angle$ rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\angle$ deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 7 Find Trigonometric Values Using the Unit Circle

B. **Find the exact value of $\cos\frac{\pi}{3}$. If undefined, write *undefined*.**

7B Exact Value $\cos\frac{\pi}{3}$

ref $\angle : \frac{\pi}{3}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

EXAMPLE 7 Find Trigonometric Values Using the Unit Circle

C. Find the exact value of $\tan\frac{4\pi}{3}$. If undefined, write *undefined*.

7c $\tan\frac{4\pi}{3}$

$-\tan\frac{\pi}{3}$

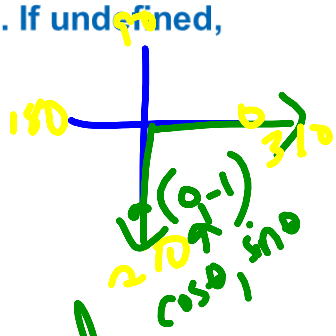
$-\sqrt{3}$

EXAMPLE 7 Find Trigonometric Values Using the Unit Circle

D. Find the exact value of sec 270°. If undefined, write *undefined*.

$\sec\theta = \frac{1}{\cos\theta}$

7d. $\sec 270^\circ$

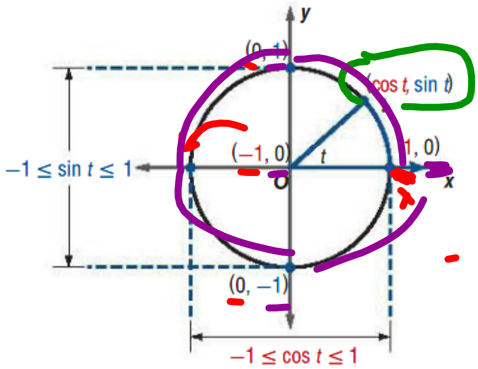


$$\frac{1}{\cos 270^\circ}$$

$$\frac{1}{0} = \text{undefined}$$

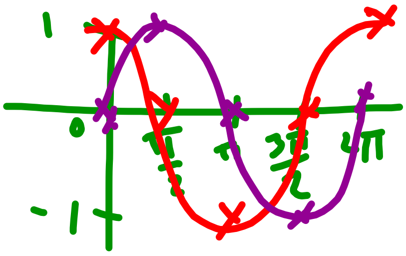
As defined by wrapping the number line around the unit circle, the domain of both the sine and cosine functions is the set of all real numbers $(-\infty, \infty)$. Extending infinitely in either direction, the number line can be wrapped multiple times around the unit circle, mapping more than one t-value to the same point $P(x, y)$ with each wrapping, positive or negative.

(cos t, sin t)



StudyTip

Radians vs. Degrees While we could also discuss one wrapping as corresponding to an angle measure of 360°, this measure is not related to a distance. On the unit circle, one wrapping corresponds to both the angle measuring 2π and the distance 2π around the circle.



Because $\cos t = x$, $\sin t = y$, and one wrapping corresponds to a distance of 2π,

$$\cos(t + 2n\pi) = \cos t \quad \text{and} \quad \sin(t + 2n\pi) = \sin t,$$

for any integer n and real number t.

The values for the sine and cosine function therefore lie in the interval $[-1, 1]$ and repeat for every integer multiple of 2π on the number line. Functions with values that repeat at regular intervals are called **periodic functions**.

$\sin t = \sin(t + 2\pi)$
period 2π

$\cos t = \cos(t + 2\pi)$
period 2π

Key Concept **Periodic Functions**

A function $y = f(t)$ is periodic if there exists a positive real number c such that $f(t + c) = f(t)$ for all values of t in the domain of f.

The smallest number c for which f is periodic is called the **period** of f.

The sine and cosine functions are periodic, repeating values after 2π, so these functions have a period of 2π. It can be shown that the values of the tangent function repeat after a distance of π on the number line, so the tangent function has a period of π and

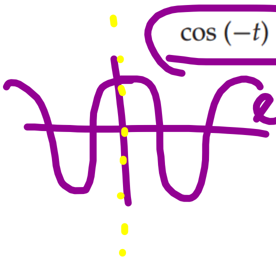
$$\tan t = \tan(t + n\pi),$$

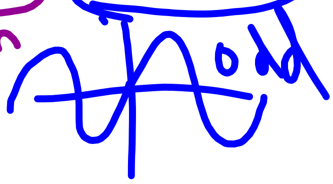
period π

for any integer n and real number t, unless both $\tan t$ and $\tan(t + n\pi)$ are undefined. You can use the periodic nature of the sine, cosine, and tangent functions to evaluate these functions.

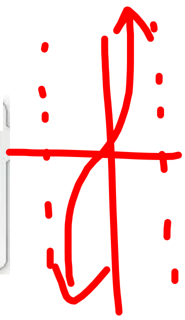
Recall from Lesson 1-2 that a function f is *even* if for every x in the domain of f, $f(-x) = f(x)$ and *odd* if for every x in the domain of f, $f(-x) = -f(x)$. You can use the unit circle to verify that the cosine function is even and that the sine and tangent functions are odd. That is,

$$\cos(-t) = \cos t \qquad \sin(-t) = -\sin t \qquad \tan(-t) = -\tan t.$$


even


odd

odd

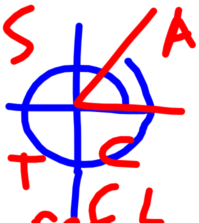


Study Tip
Periodic Functions The other three circular functions are also periodic. The periods of these functions will be discussed in Lesson 4-5.

EXAMPLE 8 Use the Periodic Nature of Circular Functions

A. Find the exact value of $\cos \frac{9\pi}{4}$.

8a Exact Value $\cos \frac{9\pi}{4}$



ref ∠ $\frac{\pi}{4}$ pos

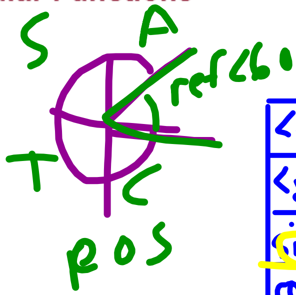
$$\cos \frac{9\pi}{4} = +\cos \frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}$$

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

EXAMPLE 8 Use the Periodic Nature of Circular Functions

B. Find the exact value of sin(–300°).



∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef

8b $\sin(-300^\circ)$

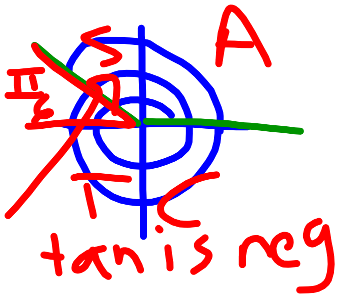
$+\sin 60$

$$\frac{\sqrt{3}}{2}$$

EXAMPLE 8 Use the Periodic Nature of Circular Functions

C. Find the exact value of $\tan \frac{29\pi}{6}$.

$$6\overline{)29}^{\;4}$$
$$\frac{24}{5}$$



8c $\tan \frac{29\pi}{6}$

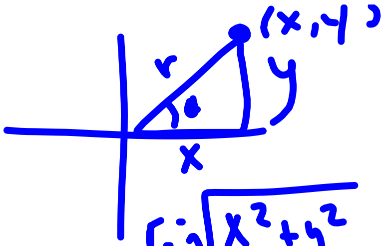
$-\tan \frac{\pi}{6}$

$-\frac{\sqrt{3}}{3}$

<rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
<deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

∠rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
∠deg	0	30	45	60	90
$\sin\theta$	0	$\frac{1}{2}$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undef

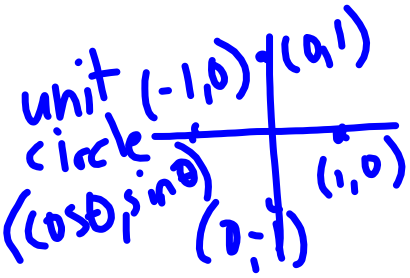
S	A
T	C



$r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$



EXAMPLE 1 Guided Practice

Let (–3, 6) be a point on the terminal side of an angle Θ in standard position. Find the exact values of the six trigonometric functions of Θ.

A. $\sin\theta = \frac{2\sqrt{5}}{5}, \cos\theta = -\frac{\sqrt{5}}{5}, \tan\theta = -2, \csc\theta = \frac{\sqrt{5}}{2}, \sec\theta = -\sqrt{5}, \cot\theta = -\frac{1}{2}$

B. $\sin\theta = -\frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}, \tan\theta = -\frac{1}{2}, \csc\theta = -\sqrt{5}, \sec\theta = \frac{\sqrt{5}}{2}, \cot\theta = -2$

C. $\sin\theta = \frac{-2\sqrt{5}}{5}, \cos\theta = \frac{\sqrt{5}}{5}, \tan\theta = -2, \csc\theta = -\frac{\sqrt{5}}{2}, \sec\theta = \sqrt{5}, \cot\theta = -\frac{1}{2}$

D. $\sin\theta = \frac{\sqrt{5}}{5}, \cos\theta = -\frac{2\sqrt{5}}{5}, \tan\theta = \frac{1}{2}, \csc\theta = \sqrt{5}, \sec\theta = -\frac{\sqrt{5}}{2}, \cot\theta = 2$

EXAMPLE 2 **Guided Practice**

Find the exact value of sec $\frac{3\pi}{2}$. If not defined, write *undefined.*

A. –1

B. 0

C. 1

D. undefined

EXAMPLE 3 Guided Practice

Find the reference angle for a 520° angle.

A. 20°

B. 70°

C. 160°

D. 200°

EXAMPLE 4 Guided Practice

Find the exact value of cos $\frac{4\pi}{3}$.

A. $-\frac{\sqrt{3}}{2}$

B. $-\frac{1}{2}$

C. $\frac{1}{2}$

D. $\frac{\sqrt{3}}{2}$

EXAMPLE 5 **Guided Practice**

Let csc $\theta = -3$, tan $\theta < 0$. Find the exact values of the five remaining trigonometric functions of θ.

A. $\sin\theta = \frac{2\sqrt{2}}{3}, \cos\theta = -\frac{1}{3}, \tan\theta = -2\sqrt{2}, \sec\theta = -3, \cot\theta = -\frac{\sqrt{2}}{4}$

B. $\sin\theta = \frac{1}{3}, \cos\theta = -\frac{2\sqrt{2}}{3}, \tan\theta = \frac{\sqrt{2}}{4}, \sec\theta = -\frac{3\sqrt{2}}{4}, \cot\theta = 2\sqrt{2}$

C. $\sin\theta = -\frac{1}{3}, \cos\theta = \frac{2\sqrt{2}}{3}, \tan\theta = -\frac{\sqrt{2}}{4}, \sec\theta = \frac{3\sqrt{2}}{4}, \cot\theta = -2\sqrt{2}$

D. $\sin\theta = -\frac{2\sqrt{2}}{3}, \cos\theta = \frac{1}{3}, \tan\theta = 2\sqrt{2}, \sec\theta = 3, \cot\theta = \frac{\sqrt{2}}{4}$

Real-World EXAMPLE 6 Guided Practice

CLOCK TOWER A 4-foot long minute hand on a clock on a bell tower shows a time of 15 minutes past the hour. What is the new position of the end of the minute hand relative to the pivot point at 5 minutes before the next hour?

A. 6 feet left and 3.4 feet above the pivot point

B. 3.4 feet left and 2 feet above the pivot point

C. 3.4 feet left and 6 feet above the pivot point

D. 2 feet left and 3.4 feet above the pivot point

EXAMPLE 7 **Guided Practice**

Find the exact value of tan $\frac{7\pi}{6}$. If undefined, write *undefined*.

A. $\sqrt{3}$

B. $\frac{\sqrt{3}}{3}$

C. $-\frac{\sqrt{3}}{3}$

D. $-\sqrt{3}$

EXAMPLE 8 Guided Practice

Find the exact value of $\cos \frac{19\pi}{4}$.

A. 1

B. –1

C. $\frac{\sqrt{2}}{2}$

D. $-\frac{\sqrt{2}}{2}$

EXAMPLE 1 **Guided Practice**

x y $r = \sqrt{(-3)^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$ A.

Let (–3, 6) be a point on the terminal side of an angle θ in standard position. Find the exact values of the six trigonometric functions of θ.

A. $\sin\theta = \frac{2\sqrt{5}}{5}, \cos\theta = -\frac{\sqrt{5}}{5}, \tan\theta = -2, \csc\theta = \frac{\sqrt{5}}{2}, \sec\theta = -\sqrt{5}, \cot\theta = -\frac{1}{2}$

B. $\sin\theta = -\frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}, \tan\theta = -\frac{1}{2}, \csc\theta = -\sqrt{5}, \sec\theta = \frac{\sqrt{5}}{2}, \cot\theta = -2$

C. $\sin\theta = \frac{-2\sqrt{5}}{5}, \cos\theta = \frac{\sqrt{5}}{5}, \tan\theta = -2, \csc\theta = -\frac{\sqrt{5}}{2}, \sec\theta = \sqrt{5}, \cot\theta = -\frac{1}{2}$

D. $\sin\theta = \frac{\sqrt{5}}{5}, \cos\theta = -\frac{2\sqrt{5}}{5}, \tan\theta = \frac{1}{2}, \csc\theta = \sqrt{5}, \sec\theta = -\frac{\sqrt{5}}{2}, \cot\theta = 2$

$\sin\theta = \frac{y}{r} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

$\cos\theta = \frac{x}{r} = \frac{-3}{3\sqrt{5}} = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$

$\tan\theta = \frac{y}{x} = \frac{6}{-3} = -2$

$\cot\theta = -\frac{1}{2}$ $\sec\theta = -\sqrt{5}$

$\csc\theta = \frac{\sqrt{5}}{2}$

EXAMPLE 2 **Guided Practice**

Find the exact value of sec $\frac{3\pi}{2}$. If not defined, write *undefined.*

A. –1

B. 0

C. 1

D. undefined

D.

$\sec\theta = \frac{1}{\cos\theta}$

$(0, -1)$

$\cos\theta \quad \sin\theta$

$\sec\frac{3\pi}{2} = \frac{1}{\cos\frac{3\pi}{2}} = \frac{1}{0} = \text{undef}$

EXAMPLE 3 **Guided Practice**

Find the reference angle for a 520° angle.

$$\begin{array}{r} 520 \\ -\,360 \\ \hline 160 \end{array}$$

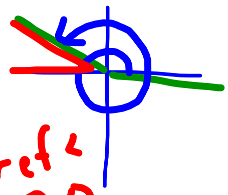
A. 20°

B. 70°

C. 160°

D. 200°

A.



ref ∠
20

EXAMPLE 4 **Guided Practice**

Find the exact value of $\cos \frac{4\pi}{3}$.

A. $-\frac{\sqrt{3}}{2}$

B. $-\frac{1}{2}$

C. $\frac{1}{2}$

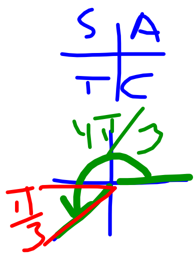
D. $\frac{\sqrt{3}}{2}$

B.

$\cos \frac{4\pi}{3}$

$-\cos \frac{\pi}{3}$

$-\frac{1}{2}$



EXAMPLE 5 **Guided Practice**

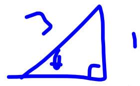
Let csc $\theta = -3$, tan $\theta < 0$. Find the exact values of the five remaining trigonometric functions of θ.

C.

$\csc\theta = \frac{1}{\sin\theta}$ $\sin\theta = \frac{1}{\csc\theta} = -\frac{1}{3}$

$\tan\theta < 0$

$\cancel{S} \mid \cancel{A}$
$T \mid C$



$\sqrt{3^2 - 1^2}$
$\sqrt{8}$
$2\sqrt{2}$

A. $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = -\frac{1}{3}$, $\tan\theta = -2\sqrt{2}$, $\sec\theta = -3$, $\cot\theta = -\frac{\sqrt{2}}{4}$

B. $\sin\theta = \frac{1}{3}$, $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\tan\theta = \frac{\sqrt{2}}{4}$, $\sec\theta = -\frac{3\sqrt{2}}{4}$, $\cot\theta = 2\sqrt{2}$

C. $\sin\theta = -\frac{1}{3}$, $\cos\theta = \frac{2\sqrt{2}}{3}$, $\tan\theta = -\frac{\sqrt{2}}{4}$, $\sec\theta = \frac{3\sqrt{2}}{4}$, $\cot\theta = -2\sqrt{2}$

D. $\sin\theta = -\frac{2\sqrt{2}}{3}$, $\cos\theta = \frac{1}{3}$, $\tan\theta = 2\sqrt{2}$, $\sec\theta = 3$, $\cot\theta = \frac{\sqrt{2}}{4}$

$\sin\theta = -\frac{1}{3}$ $\csc\theta = -3$

$\cos\theta = +\frac{2\sqrt{2}}{3}$ $\sec\theta = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$

$\tan\theta = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$ $\cot\theta = -2\sqrt{2}$

Real-World EXAMPLE 6 **Guided Practice**

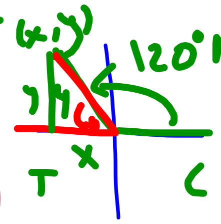
CLOCK TOWER A 4-foot long minute hand on a clock on a bell tower shows a time of 15 minutes past the hour. What is the new position of the end of the minute hand relative to the pivot point at 5 minutes before the next hour?

A. 6 feet left and 3.4 feet above the pivot point

B. 3.4 feet left and 2 feet above the pivot point

C. 3.4 feet left and 6 feet above the pivot point

D. 2 feet left and 3.4 feet above the pivot point



D.

$\sin 120 = \frac{y}{4}$

$\sin 60 = \frac{y}{4}$

$\frac{\sqrt{3}}{2} = \frac{y}{4}$

$\frac{4\sqrt{3}}{2} = y$

$2\sqrt{3} = y$

$3.4 \approx y$

$\cos 120 = \frac{x}{4}$

$-\cos 60 = \frac{x}{4}$

$-\frac{1}{2} = \frac{x}{4}$

$-2 = x$

$(-2, 3.4)$

x, y

EXAMPLE 7 **Guided Practice**

Find the exact value of $\tan \frac{7\pi}{6}$. If undefined, write *undefined*.

A. $\sqrt{3}$

B. $\frac{\sqrt{3}}{3}$

C. $-\frac{\sqrt{3}}{3}$

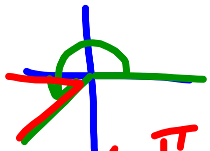
D. $-\sqrt{3}$

B.

$\tan \frac{7\pi}{6}$

$+\tan \frac{\pi}{6}$

$\frac{\sqrt{3}}{3}$



ref∠: $\frac{\pi}{6}$

S A

T C

pos

EXAMPLE 8 **Guided Practice**

Find the exact value of $\cos \frac{19\pi}{4}$.

A. 1

B. -1

C. $\frac{\sqrt{2}}{2}$

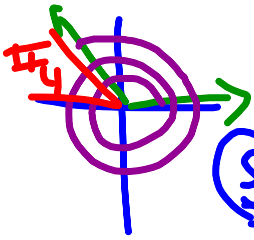
D. $-\frac{\sqrt{2}}{2}$

D.

$\cos \frac{19\pi}{4}$

$-\cos \frac{\pi}{4}$

$-\frac{\sqrt{2}}{2}$



$2\pi = \frac{8\pi}{4}$

$\frac{19\pi}{4} = \frac{8\pi}{4} + \frac{8\pi}{4} + \frac{3\pi}{4}$

S A
T C

cos is neg

4-3 Trigonometric Functions on the Unit Circle

EVEN PAGE

TOC

EQ: Can you find values of trigonometric functions for any angle, and can you find them using the unit circle?

How are you doing? Write answer next to Essential Question
1. I don't understand the material
2. I understand a little.
3. I understand this material.
4. I could teach this to someone

Summary: At least 3 sentences...