10.4 Hyperbolas

Objectives

- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- ~~Use properties of hyperbolas to solve real-life problems~~.
- Classify conics ~~from their general equations.~~

Introduction

Introduction

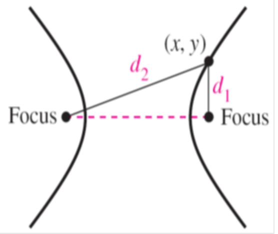
The definition of a **hyperbola** is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed.

For a hyperbola, the absolute value of the *difference* of the distances between the foci and a point on the hyperbola is fixed.

Introduction

Definition of Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane for which the absolute value of the difference of the distances from two distinct fixed points, called **foci,** is constant. See Figure 10.20.



$|d_2 - d_1|$ is a constant.

Figure 10.20

Introduction

The graph of a hyperbola has two disconnected parts called **branches**.

The line through the two foci intersects the hyperbola at two points called the **vertices**.

The line segment connecting the vertices is the **transverse axis**, and the midpoint of the transverse axis is the **center** of the hyperbola.

Introduction

Consider the hyperbola in Figure 10.21 with the following points.

Center: (h, k) Vertices: $(h \pm a, k)$ Foci: $(h \pm c, k)$

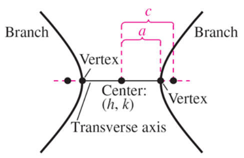
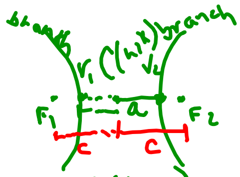


Figure 10.21

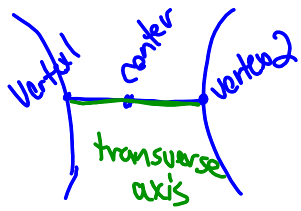
Note that the center is the midpoint of the segment joining the foci.

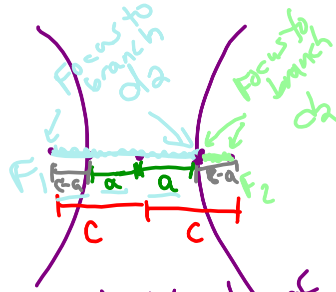


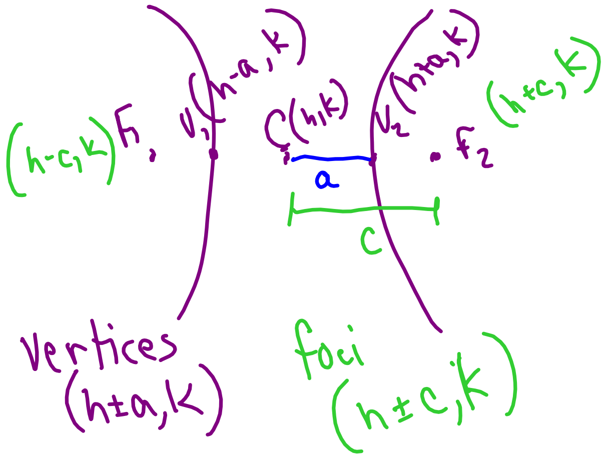
$C(h,k)$

$V_1 \& V_2$ $(h \pm a, k)$

$F_1 F_2$ $(h \pm c, k)$







point (x,y) is absolute value of difference of distances from foci to branch

$|d_2 - d_1|$

$|a + a + (c-a) - (c-a)|$

$|2a + c - a - c + a|$

$|2a|$

$2a$

Introduction

The absolute value of the difference of the distances from *any* point on the hyperbola to the two foci is constant. Using a vertex, this constant value is

$$|[2a + (c - a)] - (c - a)| = |2a|$$

$$= 2a$$ Length of transverse axis

or simply the length of the transverse axis. Now, if you let (x, y) be *any* point on the hyperbola, then

$$|d_2 - d_1| = 2a$$

Introduction

You would obtain the same result for a hyperbola with a vertical transverse axis.

The development of the standard form of the equation of a hyperbola is similar to that of an ellipse.

Introduction

Note in the definition below the *a*, *b*, and *c* are related differently for hyperbolas than for ellipses.

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin, then the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis is vertical.}$$

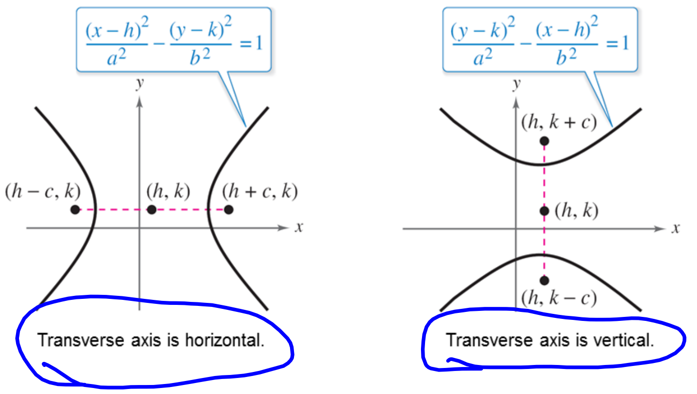
vertices $(h \pm a, k)$ foci $(h \pm c, k)$

vertices $(h, k \pm a)$ foci $(h, k \pm c)$

$c^2 = a^2 + b^2$

Introduction

Both the horizontal and vertical orientations for a hyperbola are shown below.



Example 1 – *Finding the Standard Equation of a Hyperbola*

1. Find the standard form of the equation of the hyperbola with foci (–1, 2) and (5, 2) and vertices (0, 2) and (4, 2).

and we will graph it as well.

center (midpt foci or vert)

$C = \left(\frac{-1+5}{2}, \frac{2+2}{2}\right)$

$C = (2, 2)$

(h, k)

a = dist center to vertex

$a = 4 - 2 = 2$

c = dist center to focus

$c = 5 - 2 = 3$

$c^2 = a^2 + b^2$

$3^2 = 2^2 + b^2$

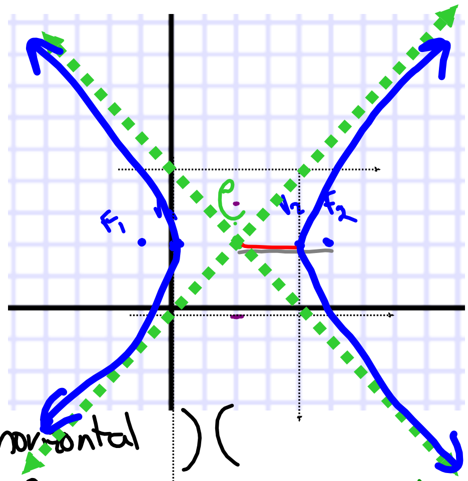
$9 - 4 = b^2$

$\sqrt{5} = b$

$(h, k \pm b)$

$(2, 2 \pm \sqrt{5})$

$(2, 4.2)$ $(2, -.2)$



Transverse axis is horizontal)(

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{(\sqrt{5})^2} = 1$$

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1$$

To Graph using box method

- make box with dimension 2a by 2b centered at (h, k)
- draw asymptotes through corners
- draw hyperbola

Example 1 – *Finding the Standard Equation of a Hyperbola*

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$C: (2, 2)$
(h, k)

a = dist center to vertex

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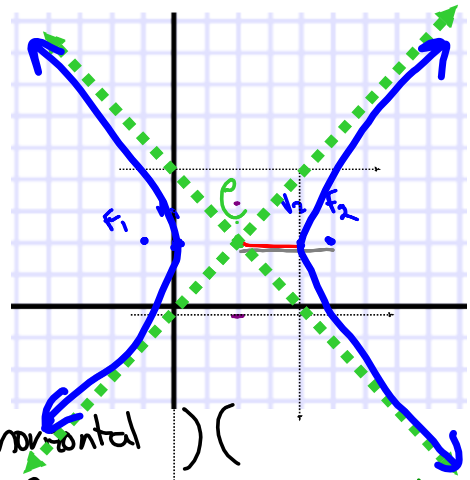

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$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

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To Graph using box method

- make box with dimension 2a by 2b centered at (h, k)
- draw asymptotes through corners
- draw hyperbola

$C: (2, 2)$
h, k

Find asymptotes by formula

$$y = k \pm \frac{b}{a}(x - h)$$

$y = 2 \pm \frac{\sqrt{5}}{2}(x - 2)$

$y = 2 + \frac{\sqrt{5}}{2}(x - 2)$ $\quad$ $y = 2 - \frac{\sqrt{5}}{2}(x - 2)$

$y = \frac{\sqrt{5}}{2}x + 2 - \sqrt{5}$ $\quad$ $y = -\frac{\sqrt{5}}{2}x + 2 + \sqrt{5}$

$y \approx 1.1x - .2$ $\quad$ $y \approx -1.1x + 4.2$

Asymptotes of a Hyperbola

Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola, as shown in Figure 10.23.

The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k).

The line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ [or $(h + b, k)$ and $(h - b, k)$] is the **conjugate axis** of the hyperbola.

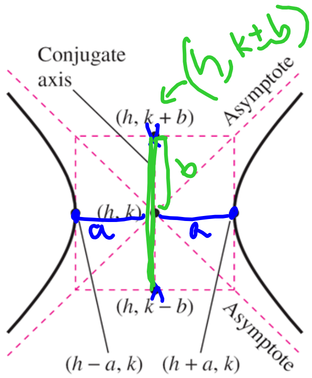


Figure 10.23

Asymptotes of a Hyperbola

Asymptotes of a Hyperbola

The equations of the asymptotes of a hyperbola are

$$y = k \pm \frac{b}{a}(x - h)$$ Transverse axis is horizontal.

$$y = k \pm \frac{a}{b}(x - h).$$ Transverse axis is vertical.

Example 2 – *Using Asymptotes to Sketch a Hyperbola*

2. Sketch the hyperbola $4x^2 - y^2 = 16$.

(a) $\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$

$\frac{x^2}{4} - \frac{y^2}{16} = 1$ Transverse axis is horiz)(

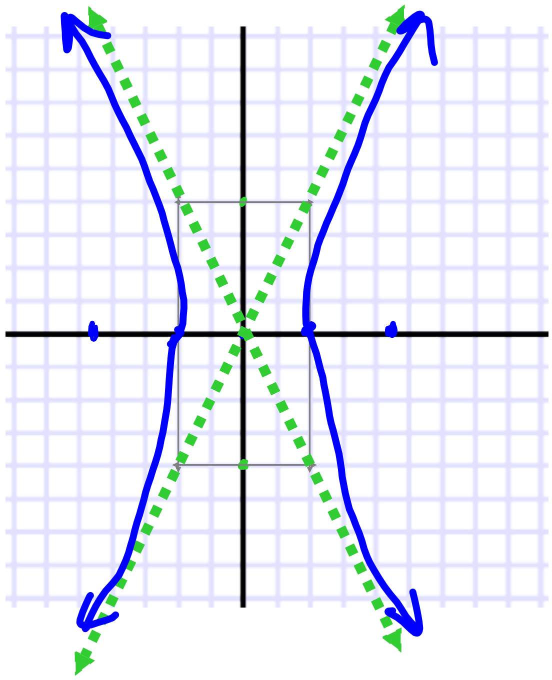
$a^2 = 4$ $b^2 = 16$

$a = 2$ $b = 4$

Center (0,0)
h, k

vertices $(h \pm a, k)$
$(0 \pm 2, 0)$
$(2,0)$ $(-2,0)$

endpt conjugate axis $(h, k \pm b)$
$(0, 0 \pm 4)$
$(0,4)$ $(0,-4)$



- make rectangle
- draw asymptotes
- draw hyperbola
- Find & plot foci

$c^2 = a^2 + b^2$

$c = \sqrt{2^2 + 4^2}$

$c = \sqrt{20}$

$c = 2\sqrt{5}$

Foci
$(h \pm c, k)$
$(0 + 2\sqrt{5}, 0)$ $(0 - 2\sqrt{5}, 0)$
$(2\sqrt{5}, 0)$, $(-2\sqrt{5}, 0)$
$(4.5, 0)$ $(-4.5, 0)$

Example 2 – *Solution*

cont'd

Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure 10.25.

Note that the asymptotes are $y = 2x$ and $y = -2x$.

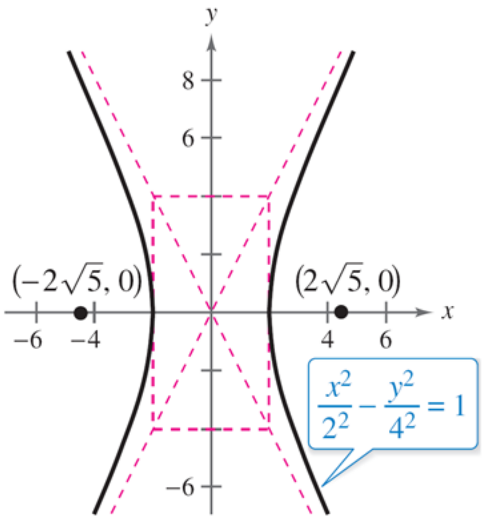
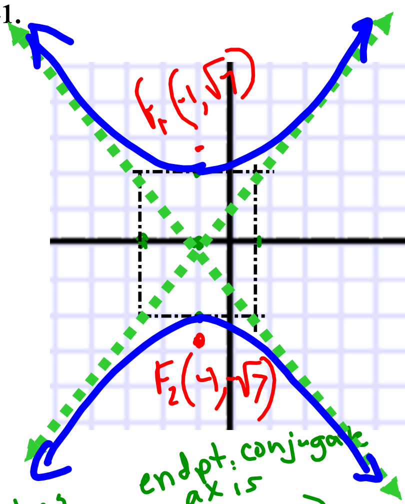


Figure 10.25

Finding the Asymptotes of a Hyperbola

Example 3 Sketch the hyperbola $4x^2 - 3y^2 + 8x + 16 = 0$.

Find the equations of its asymptotes and find the foci.



③ $4x^2 - 3y^2 + 8x + 16 = 0$

$4x^2 + 8x + \_ - 3y^2 = -16 + \_$

$4(x^2 + 2x + 1) - 3y^2 = -16 + 4(1)$

$\frac{4(x+1)^2}{-12} - \frac{3y^2}{-12} = \frac{-12}{-12}$

$\frac{(x+1)^2}{-3} + \frac{y^2}{4} = 1$

$\frac{y^2}{4} - \frac{(x+1)^2}{3} = 1$

Transverse axis is vertical

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

center (h,k)
$(-1,0)$

vertices
$(h, k \pm a)$
$(-1, 0 \pm 2)$
$(-1,-2), (-1,2)$

end pt. conjugate axis
$(h \pm b, k)$
$(1 \pm \sqrt{3}, 0)$
$(1+\sqrt{3}, 0)$ $(1-\sqrt{3}, 0)$
$(.7, 0)$ $(-2.7, 0)$

$a = 2$ $b = \sqrt{3}$

find & graph foci $(h, k \pm c)$

$c^2 = a^2 + b^2$

$c = \sqrt{2^2 + (\sqrt{3})^2}$

$c = \sqrt{7}$

$(-1, 0+\sqrt{7})$ $(-1, 0-\sqrt{7})$
$(-1, \sqrt{7})$ $(-1, -\sqrt{7})$
$(-1, 2.7)$ $(-1, -2.7)$

Equation of asymptotes

$y = k \pm \frac{a}{b}(x-h)$

$y = 0 \pm \frac{2}{\sqrt{3}}(x - -1)$

$y = \pm \frac{2\sqrt{3}}{3}(x+1)$

$y = \frac{2\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}$

$y \approx 1.2x + 1.2$

$y = -\frac{2\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}$

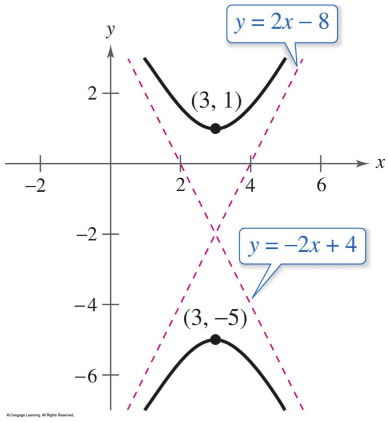
$y \approx -1.2x - 1.2$

Optional problem - we are not doing this one together.

Using Asymptotes to find the Standard Equation

Example 4 Find the standard form of the equation of the hyperbola having vertices (3, -5) and (3, 1) and having asymptotes y = 2x - 8 and y = -2x + 4 as shown in Figure 10.27

Figure 10.27



Asymptotes of a Hyperbola

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a}$$

Eccentricity

and because $c > a$, it follows that $e > 1$. When the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 10.28.

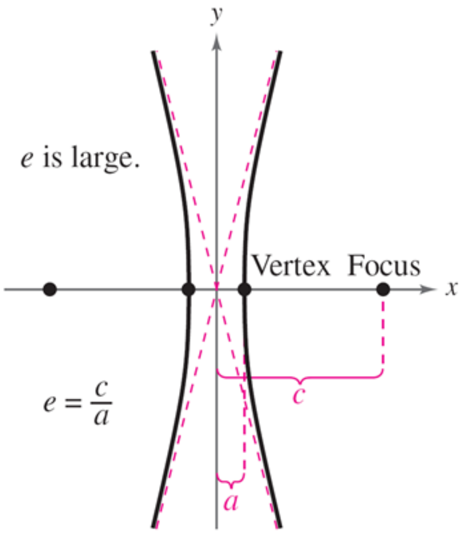


Figure 10.28

Asymptotes of a Hyperbola

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

When the eccentricity is close to 1, the branches of the hyperbola are more narrow, as shown in Figure 10.29.

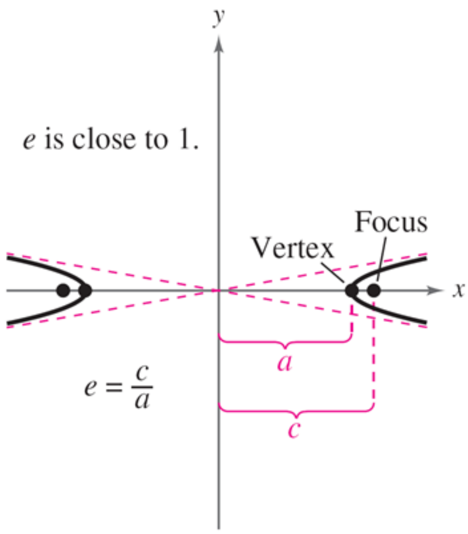


Figure 10.29

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Applications

Example 5 – *An Application Involving Hyperbolas*

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

5. Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B.

Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

Example 5 – *An Application Involving Hyperbolas*

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B.

Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

Solution:

Begin by representing the situation in a coordinate plane. The distance between the microphones is 1 mile, or 5280 feet.

Example 5 – *Solution* cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

So, position the point representing microphone A 2640 units to the right of the origin and the point representing microphone B 2640 units to the left of the origin, as shown in Figure 10.30.

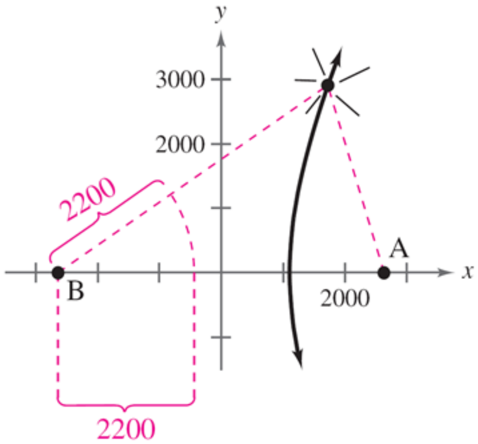


Figure 10.30

Example 5 – *Solution*

cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Assuming sound travels at 1100 feet per second, the explosion took place 2200 feet farther from B than from A.

The locus of all points that are 2200 feet closer to A than to B is one branch of a hyperbola with foci at A and B.

Because the hyperbola is centered at the origin and has a horizontal transverse axis, the standard form of its equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Example 5 – *Solution*

cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Because the foci are 2640 units from the center, $c = 2640$.

Let d_A and d_B be the distances of any point on the hyperbola from the foci at A and B, respectively. You have

$$|d_B - d_A| = 2a$$

$$|2200| = 2a$$ The points are 2200 feet closer to A than to B.

$$1100 = a.$$ Divide each side by 2.

Example 5 – *Solution*

cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

So,

$$b^2 = c^2 - a^2$$

$$= 2640^2 - 1100^2$$

$$= 5{,}759{,}600,$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

Applications

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Another interesting application of conic sections involves the orbits of comets in our solar system.

Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits.

The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 10.31.

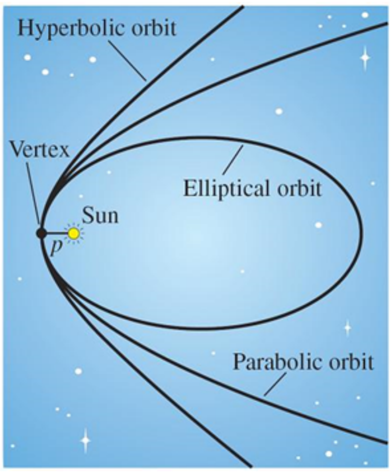


Figure 10.31

Applications

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Undoubtedly, there have been many comets with parabolic or hyperbolic orbits that were not identified. We only get to see such comets *once*.

Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

Applications

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

If p is the distance between the vertex and the focus (in meters), and v is the velocity of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$

2. Parabola: $v = \sqrt{2GM/p}$

3. Hyperbola: $v > \sqrt{2GM/p}$

In each of these relations, $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

General Equations of Conics

General Equations of Conics

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. *Circle:* $A = C$ — $A \neq 0$
2. *Parabola:* $AC = 0$ — $A = 0$ or $C = 0$, but not both.
3. *Ellipse:* $AC > 0$ — A and C have like signs.
4. *Hyperbola:* $AC < 0$ — A and C have unlike signs.

The test above is valid when the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, whose graph is not a conic.

Example 6 – *Classifying Conics from General Equations*

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

6. Classify the graph of each equation.

a. $4x^2 - 9x + y - 5 = 0$

b. $4x^2 - y^2 + 8x - 6y + 4 = 0$

Example 6 – *Classifying Conics from General Equations*

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

6. Classify the graph of each equation.

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$AC = 4(0) = 0.$ Parabola

So, the graph is a parabola.

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$AC = 4(-1) < 0.$ Hyperbola

So, the graph is a hyperbola.

Example 6 – *Classifying Conics from General Equations* cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

c. $2x^2 + 4y^2 - 4x + 12y = 0$

d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Example 6 – *Classifying Conics from General Equations* cont'd

We are not doing this, but you should look it over - so you are familiar with the concept in the future.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$AC = 2(4) > 0$. Ellipse

So, the graph is an ellipse.

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$A = C = 2$. Circle

So, the graph is a circle.

10.4 Hyperbolas

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