10.3 Ellipses

Objectives

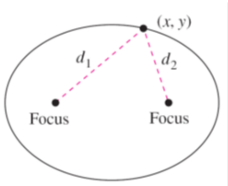
- Write equations of ellipses in standard form and graph ellipses.
- ~~Use properties of ellipses to model and solve real-life problems~~.
- ~~Find eccentricities of ellipses~~.

Introduction

Introduction

The second type of conic is called an **ellipse**. It is defined as follows.

Definition of Ellipse
An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points, called **foci,** is constant. See Figure 10.12.



$d_1 + d_2$ is constant.

Figure 10.12

Introduction

The line through the foci intersects the ellipse at two points called **vertices**.

The chord joining the vertices is the **major axis**, and its midpoint is the **center** of the ellipse.

The chord perpendicular to the ~~major axis~~ at the center is the **minor axis** of the ellipse.
See Figure 10.13.

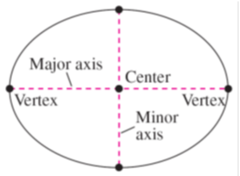
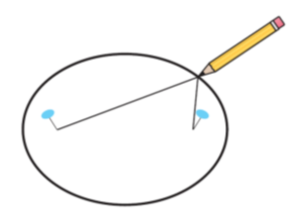


Figure 10.13

Introduction

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown below.

If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, then the path traced by the pencil will be an ellipse.



Introduction

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 10.14 with the following points.

center: (h, k) vertices: $(h \pm a, k)$ foci: $(h \pm c, k)$

Note that the center is the midpoint of the segment joining the foci.

The sum of the distances from any point on the ellipse to the two foci is constant.

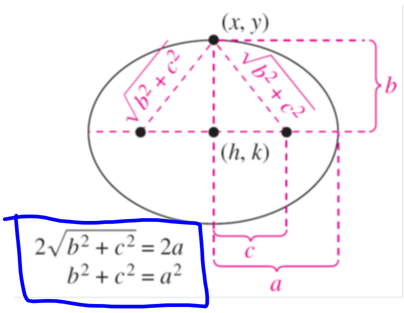
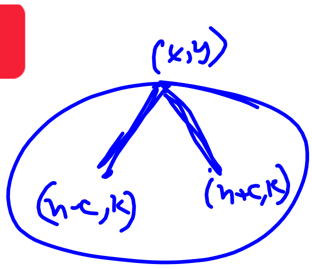


Figure 10.14

So $(a+c)+(a-c)=2a$ ← length of the major axis.

Introduction



Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a$$

Length of major axis

or simply the length of the major axis.

Now, if you let (*x*, *y*) be *any* point on the ellipse, then the sum of the distances between (*x*, *y*) and the two foci must also be 2*a*.

$$\sqrt{(x-(h-c))^2+(y-k)^2} + \sqrt{(x-(h+c))^2+(y-k)^2} = 2a$$

Introduction

That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a$$

which, after expanding and regrouping, reduces to

$$(a^2-c^2)(x-h)^2+a^2(y-k)^2=a^2(a^2-c^2).$$

$$2\sqrt{b^2+c^2}=2a$$
$$\sqrt{b^2+c^2}=a$$
$$b^2+c^2=a^2$$
$$b^2=a^2-c^2$$

$$\frac{\cancel{b^2}(x-h)^2}{a^2\cancel{b^2}}+\frac{\cancel{a^2}(y-k)^2}{\cancel{a^2}b^2}=\frac{a^2b^2}{a^2b^2}$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Equation of Ellipse

Introduction

Finally, in Figure 10.14, you can see that

$$b^2 = a^2 - c^2$$

which implies that the equation of the ellipse is

$$b^2(x - h)^2 + a^2(y - k)^2 = a^2b^2$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis.

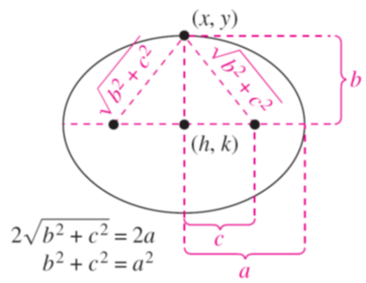


Figure 10.14

Introduction

Both results are summarized as follows.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1} \qquad \text{Major axis is horizontal.}$$

bigger

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with

$$c^2 = a^2 - b^2.$$

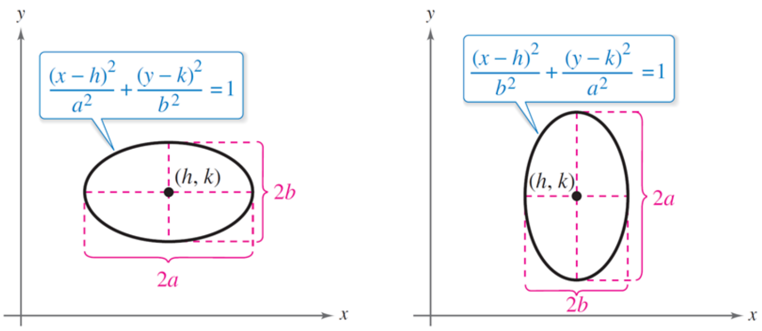
If the center is at the origin, then the equation takes one of the following forms.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \text{Major axis is vertical.}$$

Introduction

Both the horizontal and vertical orientations for an ellipse are shown below.



Major axis is horizontal.

Major axis is vertical.

Example 1 – *Finding the Standard Equation of an Ellipse*

1. Find the standard form of the equation of the ellipse having foci (0, 1) and (4, 1) and a major axis of length 6, as shown in Figure 10.15.

Center (2,1)
h,k

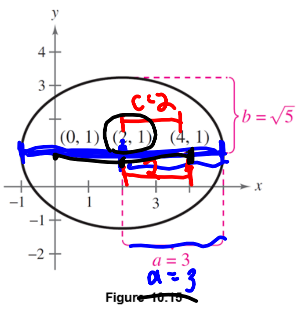


Figure 10.15

horizontal

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-2)^2}{3^2}+\frac{(y-1)^2}{(\sqrt{5})^2}=1$$

$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{5}=1$$

$c^2=a^2-b^2$

$2^2=3^2-b^2$

$4=9-b^2$

$b^2=9-4$

$b^2=5$

$b=\pm\sqrt{5}=\sqrt{5}$

Sketching an Ellipse

Example 2 Find the center, vertices, and foci of the ellipse

$x^2 + 4y^2 + 6x - 8y + 9 = 0$. Then sketch the ellipse.

② $x^2 + 4y^2 + 6x - 8y + 9 = 0$

$x^2 + 6x + \_\_ + 4y^2 - 8y + \_ = -9 + \_ + \_\_$

$x^2 + 6x + 9 + 4(y^2 - 2y + 1) = -9 + 9 + 4(1)$

$\frac{(x+3)^2}{4} + \frac{\cancel{4}(y-1)^2}{\cancel{4}} = \frac{4}{4}$

$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{1} = 1$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

bigger → major axis horiz

Center $(-3, 1)$

$a^2 = 4$

$a = 2$

vertices $(h+a, k), (h-a, k)$

$(-3+2, 1), (-3-2, 1)$

$(-1, 1), (-5, 1)$

$b^2 = 1$

$b = 1$

up & down from center

$(h, k+b), (h, k-b)$

$(-3, 1+1), (-3, 1-1)$

$(-3, 2), (-3, 0)$

foci $c^2 = a^2 - b^2$

$c^2 = 2^2 - 1^2$

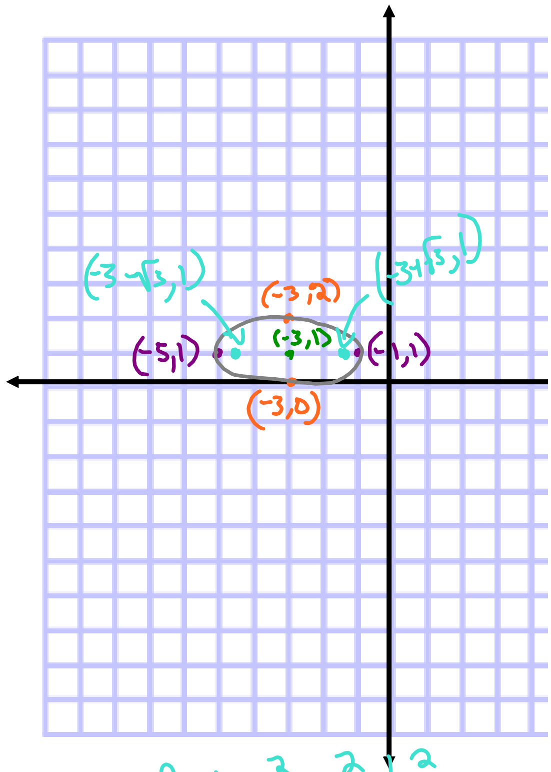
$c^2 = 3$

$c = \sqrt{3}$

$(h+c, k), (h-c, k)$

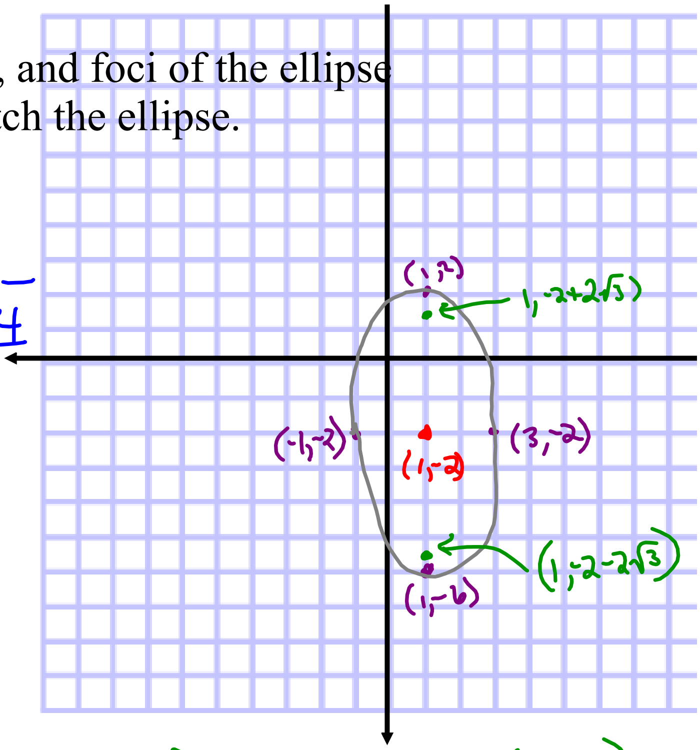
foci → $(-3+\sqrt{3}, 1), (-3-\sqrt{3}, 1)$

plot $(-1.3, 1), (-4.7, 1)$



Sketching an Ellipse

Example 3 Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$. Then sketch the ellipse.



③ $4x^2+y^2-8x+4y-8=0$

$4x^2-8x+\_\_+y^2+4y+\_\_=8+\_\_+\_\_$

$4(x^2-2x+\underline{1})+y^2+4y+\underline{4}=8+4(\underline{1})+\underline{4}$

$\frac{4(x-1)^2}{16}+\frac{(y+2)^2}{16}=\frac{16}{16}$

$\frac{(x-1)^2}{4}+\frac{(y\overset{--2}{+2})^2}{16}=1$

bigger major axis is vertical $16=a^2$

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

Center (h,k)

$(1,-2)$

Vertices $a^2=16$, $a=4$

$(h,k+a),(h,k-a)$

$(1,-2+4),(1,-2-4)$

$(1,2),(1,-6)$

$b^2=4$, $b=2$

$(h+b,k),(h-b,k)$

$(1+2,-2),(1-2,-2)$

$(3,-2),(-1,-2)$

foci

$c^2=a^2-b^2$

$c^2=4^2-2^2$

$c^2=12$

$c=\sqrt{12}$

$c=2\sqrt{3}$

$(h,k+c),(h,k-c)$

$(1,-2+2\sqrt{3}),(1,-2-2\sqrt{3})$

dec to graph $(1,1.5),(1,-5.5)$

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